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S-MATRIX CALCULATION OF THE TRIPLE-ALPHA REACTION

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ABSTRACT

We report a nonperturbative *S*-matrix calculation of the triple-α reaction. Our approach is similar to the calculation of three-body reactions in atomic physics, except that we use experimental values for the widths of the nuclear states rather than calculating them from first principles. The *S*-matrix approach has several advantages. First, we are able to define precisely the meaning of the energy-dependent widths $\Gamma(E)$ of the $\alpha + \alpha \rightarrow {}^8\mathrm{Be}$ and $\alpha + {}^8\mathrm{Be} \rightarrow {}^{12}\mathrm{C}$ reactions in terms of the interaction Hamiltonian for these reactions. Second, we show that *at resonance the width* $\Gamma(E)$ *is independent of the electron screening.* Third, we are able to give an exact expression for the abundance of (unstable) ^{8}Be compound nuclei for all temperatures and densities (not just for high temperatures where the Saha approach is valid). Finally, we note that our *S*-matrix formalism can be applied to other reactions in which electron screening is important, including three-body reactions and reactions involving weak interactions.

We obtain an *analytical expression* for the screened triple-α reaction rate which is accurate for all temperatures and densities. We identify a new nonresonant reaction regime at high ($\log \rho > 9.7$) densities. We compare our results with those of Cameron and of Nomoto, Thielemann, and Miyaji, and verify the latter's expression for the unscreened reaction rate. However, we show that the reaction rate in the pycnonuclear regime cannot be obtained from the unscreened rate using a screening factor, and the results of Nomoto, Thielemann, and Miyaji therefore cannot be used in this regime.

Subject headings: nuclear reactions — nucleosynthesis — stars: abundances — supernovae — X-rays: bursts

I. INTRODUCTION

The triple-α reaction is one of the most famous nuclear reactions in astrophysics. It is central to the idea that the heavy elements are formed by nuclear processing in stars during their late stages of evolution. This idea initially encountered a stumbling block in that no stable nuclei exist for atomic numbers $A = 5$ or 8; thus neither proton-capture nor α-capture reactions on ^{4}He can lead to heavier nuclei. This led to the demonstration by Salpeter (1952, 1953) that ^{12}C could be formed in the collision of three α-particles, the conjecture by Hoyle (1954) that this process must be enhanced by a hitherto unknown resonance of ^{12}C, and the experimental discovery by Fowler and coworkers of just such a resonance (Fowler and Greenstein 1956; Cook *et al.* 1957; see also Salpeter 1957). Such helium burning is now recognized as one of the major nuclear burning stages in stellar evolution (cf. Clayton 1968). It occurs in the helium cores of normal stars at temperatures $T \sim 10^8$ K and densities $\rho \sim 10^2$–10^5 g cm^{-3}.

More recently, the triple-α reaction has become recognized as important in the very late stages of stellar evolution. It is of particular importance in accretion onto degenerate dwarfs and neutron stars. During such accretion, a helium layer is built up either by hydrogen shell burning in the accreted material at the surface of the star, or by transfer of material from a companion star which has evolved to become a helium main-sequence star or a degenerate dwarf. If the accreting star is a degenerate dwarf, the triple-α reaction determines the conditions under which a helium detonation or carbon deflagration Type I supernova may occur (Nomoto 1982). If it is a neutron star, the triple-α reaction determines the properties of X-ray bursts (see the reviews by Lewin and Joss 1983 and Taam 1985; see also Miyaji and Nomoto 1985) and, possibly, γ-ray bursts (see the reviews by Woosley 1984; Hameury *et al.* 1984). In either case, helium burning takes place at low temperatures ($T \sim 10^7$–10^8 K) and high densities ($\rho \sim 10^8$–10^9 g cm^{-3}), as a result of the high surface gravity of the accreting star. Under such conditions, the electron background is highly degenerate and partially screens the Coulomb forces between the reacting nuclei.

Cameron (1959) carried out the first calculations of the triple-α reaction at low temperatures and high densities. He found that electron screening dramatically enhances the nuclear reaction rate (see also Clayton 1968; Salpeter and Van Horn 1969). Since then a controversy has developed about how to treat the electron screening properly, given the three-body nature of the reaction and the unstable intermediate state consisting of the ^{8}Be compound nucleus. Recently, Nomoto, Thielemann, and Miyaji (1985, hereafter NTM) have carefully examined the reaction, using a heuristic approach to treat the unstable ^{8}Be intermediate state in the nonresonant regime.

In this paper, we report a nonperturbative *S*-matrix calculation of the triple-α reaction. Our approach is similar to the calculation of three-body reactions in atomic physics (Heitler 1954; Lamb and ter Haar 1971), except that we use experimental values for the widths of the nuclear states rather than calculating them from first principles. The *S*-matrix approach has several advantages. First, we are able to define precisely the meaning of the energy-dependent widths $\Gamma(E)$ of the $\alpha + \alpha \rightarrow {}^8\mathrm{Be}$ and $\alpha + {}^8\mathrm{Be} \rightarrow {}^{12}\mathrm{C}$ reactions in terms of the interaction Hamiltonian for these reactions. Second, we show that *at resonance the width* $\Gamma(E)$ *is independent of the* *electron screening*. Third, we are able to give an exact expression for the abundance of (unstable) ^{8}Be compound nuclei for all temperatures and densities (not just for high temperatures where the Saha approach is valid). Finally, we note that our S-matrix formalism can be applied to other reactions in which electron screening is important, including three-body reactions and reactions involving weak interactions.

We obtain the following results. We present an *analytical expression* for the screened triple-α reaction rate which is accurate for all temperatures and densities. We identify a new nonresonant reaction regime at high ($\log \rho > 9.7$) densities. We compare our results with those of Cameron (1959) and NTM, and verify NTM's expression for the unscreened reaction rate. However, we show that the reaction rate in the pycnonuclear regime cannot be obtained from the unscreened rate using a screening factor, and that NTM's results therefore cannot be used in this regime.

We derive the S-matrix for n-body reactions in § II, and obtain expressions for the energy widths of the intermediate states, the abundance of the unstable intermediate states, and the cross sections for the reactions involved. We apply our formalism to the triple-α reaction in § III. We discuss our results and derive an analytical expression for the triple-α reaction rate that is accurate for all temperatures and densities in § IV. In § V we compare our results with those of Cameron (1959) and NTM, and in § VI we summarize our conclusions. All of the mathematical details are left to Appendices A–E.

II. SCATTERING MATRIX PICTURE

The triple-α reaction is a two-stage process. First, two α-particles collide to form the unstable compound nucleus ^{8}Be. Before the ^{8}Be can decay, a third α-particle collides with it to form an excited state of ^{12}C, which then decays by two successive γ-ray transitions to its ground state. Hence one must be able to treat unstable intermediate states properly in order to obtain the triple-α reaction rate. Previous work by Cameron (1959) and NTM has used the following heuristic argument to treat the unstable intermediate state: the width of the unstable ^{8}Be state gives its lifetime, and therefore the formation rate of excited ^{12}C nuclei is proportional to this lifetime times the Breit-Wigner cross section for the reaction $\alpha + {}^8\mathrm{Be} \rightarrow {}^{12}\mathrm{C}^*$. However, it is impossible to define the lifetime of the unstable intermediate state for different specific energies because of the Heisenberg uncertainty principle. Thus the validity of the heuristic argument is unclear.

By starting with the Schrödinger equation, we are able to derive rigorously an expression for the reaction rate using the S-matrix, without assuming the form of the cross sections for the intermediate reactions or that the unstable intermediate state is in equilibrium.

a) S-Matrix

Suppose that we have $(k + 1)$ stable states when there is no interaction between them. When the interaction is turned on, these states are no longer stable and they undergo transitions with one another. Let us separate the total Hamiltonian into an unperturbed part, H_0, and an interaction part H_{int}. Let us further assume that the eigenstates $|n\rangle$ $(n = 0, 1, 2, \ldots, k)$ of H_0 are complete and orthogonal, with eigenvalues E_n and normalizations $\mathcal{N}_n$, and that both H_0 and H_{int} are Hermitian operators. Since the unperturbed Hamiltonian is Hermitian, the unperturbed energies E_n are real numbers. Using the interaction picture, the Schrödinger equation can be written as

$$i\hbar \mathcal{N}_n^{1/2} \dot{b}_n(t) = \sum_m H_{nm} e^{i(E_n - E_m)t/\hbar} \mathcal{N}_m^{1/2} b_m(t) , \tag{2.1}$$

where $H_{nm} = \langle n | H_{\mathrm{int}} | m \rangle / (\mathcal{N}_n \mathcal{N}_m)^{1/2}$. Here $b_n(t)$ is the amplitude of the state $e^{iH_0 t/\hbar}|n\rangle$, and $\mathcal{N}_n b_n(t)$ corresponds to the S-matrix element $\langle n | S(t; 0) | 0 \rangle$. Thus the S-matrix gives the abundance of each state as a function of time t.

The problem then reduces to solving equation (2.1). We solve it by Fourier-transforming the $b_n(t)$ to form the corresponding Green's functions (Heitler 1954; Lamb and ter Haar 1971). The transformed equations contain no derivatives of the Green's functions, and may therefore be solved simply by algebra. We give the mathematical details of the solution in Apendices A–D.

b) Saha Equation

Let us illustrate our formalism by applying it to the simplest case, in which there is one unstable intermediate state. The physical situation is as follows. The initial state undergoes transitions between the initial, the intermediate, and the final states. Under some conditions, thermal equilibrium may be established between two states, and their relative abundances may then be given by the Saha equation. The final state is produced after passing through the intermediate state. We may also expect that the cross section for the final state will take the Breit-Wigner form. In this section, we show that our formalism does lead to the Breit-Wigner cross section for the final state and that the Saha equation for the abundance of the intermediate state is obtained for special initial conditions. We also treat carefully the scattering of identical particles.

From Appendix B, the equilibrium value of the intermediate state $|m\rangle$ is given by

$$|b_m(t)|^2 = \frac{\mathcal{N}_0 |H_{m0}|^2}{(E_0 - E_m)^2 + (\tilde{\Gamma}/2)^2} , \tag{2.2}$$

where $\tilde{\Gamma}$ is the effective width (see below). The phase-space density of the relative momentum of the final state is

$$\rho(E) = \frac{4\pi V p^2}{h^3 v} . \tag{2.3}$$

The momentum p and velocity v are related to E in the Newtonian way using the reduced mass. All particles are assumed to have spin zero. $\Gamma_Q(E)$ is given by $2\pi\rho_Q(E)$ times the absolute square of the matrix element of the interaction Hamiltonian. We call this definition the standard expression for the energy width. If the channel Q consists of identical particles, the effective width $\tilde{\Gamma}_Q(E)$ is smaller than the standard expression by a factor of 2 because the available phase space is half as large. The total effective width $\tilde{\Gamma}$ is the sum of the effective widths of all the possible exit channels. The energy width of the initial state, $\Gamma_0(E)$, is defined in the same way. We choose $\mathcal{N}_0 = N_1 N_2$, where N_1 and N_2 are the total numbers of the particles 1 and 2 in a volume V, respectively. Hence we obtain

$$N_1 N_2 |H_{m0}|^2 = \pi \lambda\!\!\!^{-2} n_1 n_2 v \hbar \Gamma_0(E_{12}) V \,, \tag{2.4}$$

where the de Broglie wavelength, $\lambda\!\!\!^{-} = \hbar/p$, and the number densities $n_1 = N_1/V$ and $n_2 = N_2/V$ are used. The momentum and velocity are related to the relative energy of the initial state by the reduced mass.

Usually the initial state is not purely monochromatic. We must average over the initial energy distribution $f_0(E_0)$. The average $\{|b_m(t)|^2\}$ is defined in terms of $|b_m(t)|^2$, and the initial energy distribution by

$$\{|b_m(t)|^2\} = \int |b_m(t)|^2 f_0(E_0) dE_0 \,. \tag{2.5}$$

Since $\{|b_m(t)|^2\}$ is the average total number of the intermediate state, its average number density, $n_m(t)$, is given by dividing it by the volume V. The initial energy distribution $f_0(E_0)$ can be decomposed into the distribution of the center-of-mass energy $f_{1+2}(E_{1+2})$ and that of the relative energy $f_{12}(E_{12})$. Since we assume that $b_m(t)$ does not depend on the center-of-mass energy, the integral over E_{1+2} gives unity. For the identical initial particles, we must divide by a factor of 2. Using the effective energy widths, we obtain

$$n_m(t) = \pi \hbar n_1 n_2 \int_0^\infty \lambda\!\!\!^{-2} \frac{\tilde{\Gamma}_0(E_{12}) v f_{12}(E_{12}) dE_{12}}{(E_{12} - E_m)^2 + [\tilde{\Gamma}(E_{12})/2]^2} \,. \tag{2.6}$$

If we assume Maxwellian distributions for the initial particles, $f_{12}(E_{12})$ can be written as

$$f_{12}(E_{12}) = \frac{2}{\sqrt{\pi}} \frac{E_{12}^{1/2}}{(kT)^{3/2}} e^{-E_{12}/kT} \,. \tag{2.7}$$

The average number density then becomes

$$n_m = n_1 n_2 \left[\frac{2\pi(m_1 + m_2)\hbar^2}{m_1 m_2 kT}\right]^{3/2} \frac{1}{2\pi} \int_0^\infty \frac{\tilde{\Gamma}_0(E_{12}) e^{-E_{12}/kT} dE_{12}}{(E_{12} - E_m)^2 + [\tilde{\Gamma}(E_{12})/2]^2} \,. \tag{2.8}$$

If the effective energy width $\tilde{\Gamma}(E_m)$ is sufficiently small and $\tilde{\Gamma}_0(E_m)$ is the dominant factor in $\tilde{\Gamma}(E_m)$ at the resonant energy E_m, the denominator of the integrand behaves as a δ-function. The average number density of the intermediate state, which we denote by n_m^*, is then given by

$$\frac{n_m^*}{n_1 n_2} = \left[\frac{2\pi(m_1 + m_2)\hbar^2}{m_1 m_2 kT}\right]^{3/2} e^{-E_m/kT} \,. \tag{2.9}$$

This is the Saha equation for the relative densities of the initial and intermediate states. Thus we have demonstrated that our expression reduces to the Saha equation as a special case.

c) *Breit-Wigner Cross Section*

The quantity $d|b_Q(t)|^2/dt$ is the formation rate of Q-channel particles in volume V. When the Q-channel consists of two identical particles, one creation of the Q-channel corresponds to two creations of the formed particle. The observed creation rate should therefore be multiplied by a factor of 2. The formation rate per unit volume is then given by $r_Q(t) = (1 + \delta_{Q1,Q2})|b_Q(t)|^2$. Using the effective and standard energy widths, we obtain

$$r_Q(t) = \pi \lambda\!\!\!^{-2} n_1 n_2 \frac{\Gamma_Q(E_{12}) \Gamma_0(E_{12}) v}{(E_{12} - E_m)^2 + [\tilde{\Gamma}(E_{12})/2]^2} \,. \tag{2.10}$$

The reason why the energy width for the initial state in the numerator retains the standard value rather than the effective value is simply that we have not averaged over the energy distribution of the initial state. Equating $r_Q(t) = n_1 n_2 \sigma(E_{12}) v$ with expression (2.10), we obtain

$$\sigma(E_{12}) = \pi \lambda\!\!\!^{-2} \frac{\Gamma_Q(E_{12}) \Gamma_0(E_{12})}{(E_{12} - E_m)^2 + [\tilde{\Gamma}(E_{12})/2]^2} \,. \tag{2.11}$$

When the Q-channel does not consist of identical particles, the above cross section coincides with the Breit-Wigner formula given in Fowler, Caughlan, and Zimmerman (1967). Our definition of the cross section is consistent with the following derivation. When $r_Q(t)$ is averaged over the relative energy distribution of the initial state, which we denote by $\{r_Q(t)\}$, we must consider the identities of the particles in the initial state when calculating its energy width. Equating $\{r_Q(t)\}$ with $n_1 n_2 \langle \sigma v \rangle/(1 + \delta_{12})$, where the angle brackets denote the average over the relative energy distribution of the initial state, we obtain the same definition for the cross section. We can write down $\{r_Q(t)\}$ explicitly as

$$\{r_Q(t)\} = (1 + \delta_{Q1,Q2})\pi n_1 n_2 \int_0^\infty \bar{\lambda}^2 \frac{\tilde{\Gamma}_Q(E_{12})\tilde{\Gamma}_0(E_{12}) v f_{12}(E_{12}) dE_{12}}{(E_{12} - E_m)^2 + [\tilde{\Gamma}(E_{12})/2]^2} . \tag{2.12}$$

Under the same condition for which we derived the Saha equation, the creation rate of the channel Q can be written as

$$\{r_Q(t)\} = \frac{\Gamma_Q(E_m)}{\hbar} n_m^* . \tag{2.13}$$

The physical interpretation of this equation is as follows. The intermediate state is formed according to the Saha equation. The final-state particles are produced by the decay of the intermediate state, which has a lifetime $\hbar/\Gamma_Q(E_m)$. The standard thermonuclear triple-α reaction rate (Clayton 1968) is derived from the product of expressions like that in equation (2.13) for each stage in the reaction. Thus we have demonstrated that our expression reduces to the standard expression as a special case.

III. TRIPLE-ALPHA REACTION

a) Parameter Regimes in the (ρ, T)-Plane

The ground state of the compound nucleus ^{8}Be lies 91.78 keV above the energy of the ^{4}He+^{4}He system. This state has an α-particle decay width of 6.8 eV, and therefore usually decays quickly back into two α-particles. However, occasionally the collision of another α-particle with the ^{8}Be compound nucleus forms an excited state of ^{12}C, ^{12}C*(0^+), which lies 287.5 keV above the energy of the ^{8}Be+^{4}He system. This state almost always decays quickly back into ^{8}Be + ^{4}He or 3 ^{4}He, since the partial widths for these decays are 8.9 and 6.8 eV, respectively. However, very rarely it decays to the ground state of ^{12}C by the successive γ-ray transitions $0^+ \rightarrow 2^+$ and $2^+ \rightarrow 0^+$, with energy widths of 3.7 and 10.8 meV, respectively. We have taken the values of the energies and decay widths for the states of ^{8}Be from Ajzenberg-Selove (1984) and for the states of ^{12}C from Ajzenberg-Selove and Busch (1980); these states are shown schematically in Figure 1.

Figure 2 illustrates the resulting parameter regimes for the reaction ^{4}He + ^{4}He → ^{8}Be [the regimes for the reaction

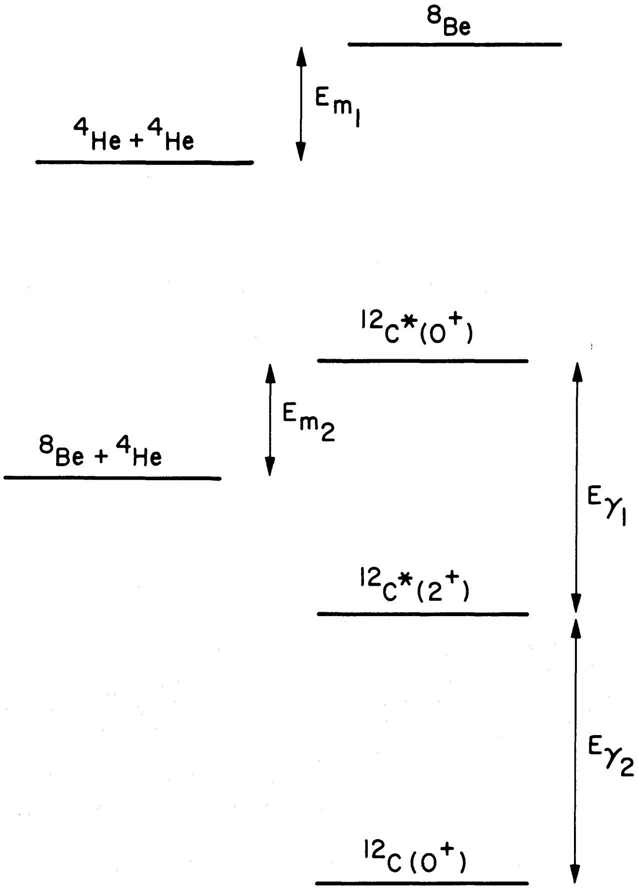

FIG. 1.—Energy level diagrams for the reactions ^{4}He + ^{4}He → ^{8}Be and ^{4}He + ^{8}Be → ^{12}C*(0^+). The quantities $\tilde{\Gamma}_{m_1}$, Γ_{m_2}, Γ_{γ_1} and Γ_{γ_2} in Table 2 are the energy widths associated with the states E_{m_1}, E_{m_2}, E_{γ_1}, and E_{γ_2}, respectively.

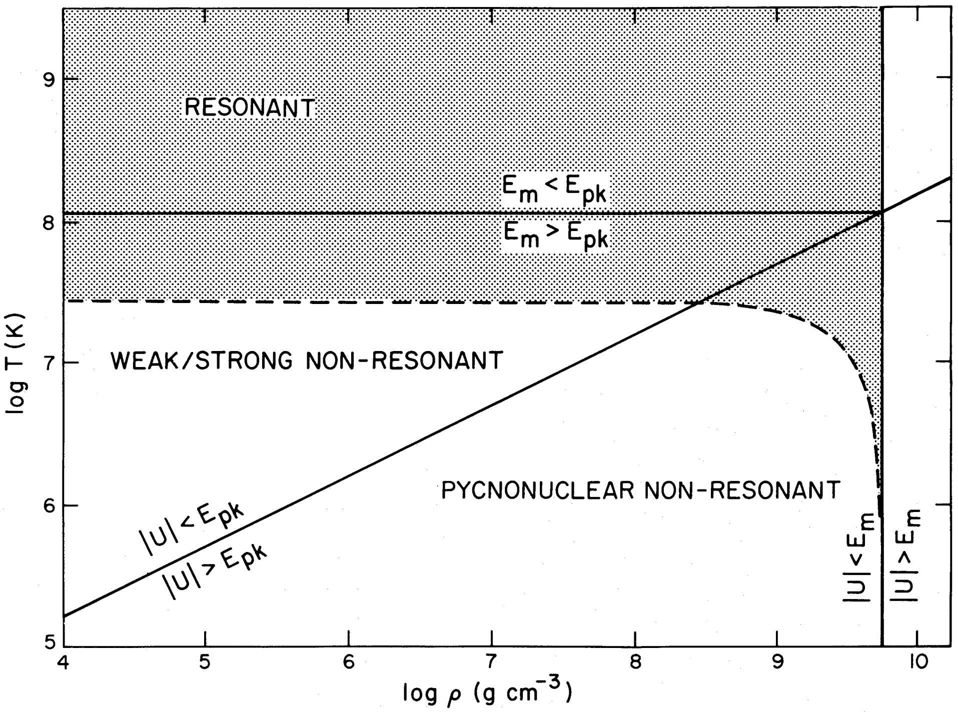

FIG. 2.—Resonant and nonresonant regimes for the ^{4}He + ^{4}He → ^{8}Be reaction assuming a pure helium composition. The quantities E_m, E_{pk}, and $|U|$ are the resonance energy (threshold energy), the Gamow peak energy, and the absolute value of the screening potential for the reaction. The shaded region at high temperature is the thermonuclear resonant regime. The weak and strong screening nonresonant regimes lie below the shaded region and to the left of the line $|U| = E_{pk}$; the pycnonuclear nonresonant regime lies below the shaded region and to the right of this line. The high-density resonant regime lies in a narrow strip just to the left of the line $|U| = E_m$. The high-density nonresonant regime lies to the right of the $|U| = E_m$ line.

^{4}He + ^{8}Be → ^{12}C*(0$^+$) are similar]. At temperatures $T \sim 10^8$ K, corresponding to the cores of normal stars, ^{4}He nuclei have kinetic energies $E \sim 100$ keV. The energy E_{pk} of the Gamow peak, which is determined by the convolution of the Coulomb barrier and the Maxwellian distributions of the reacting ^{4}He nuclei (Clayton 1968), is greater than or equal to the resonance energy E_m. Hence, the reaction proceeds via the resonance at 91.78 keV, and the abundance of ^{8}Be nuclei comes into thermal equilibrium.

When the temperature drops to $T \sim 10^7$ K, the Gamow peak energy E_{pk} falls below the resonance energy E_m (see Fig. 2), and the ^{8}Be resonance becomes inaccessible to the ^{4}He thermal distribution. However, the ^{8}Be nucleus can form by nonresonant reactions, which occur in the low-energy wing of the resonance peak. At low temperatures the rate of formation of ^{8}Be due to the nonresonant reactions dominates that due to the resonant reactions, and the abundance of ^{8}Be is no longer in thermal equilibrium.

At low temperatures and high densities, electron screening increases the effective energy of the reacting ^{4}He nuclei and enhances the reaction. Because the lower limit of the effective energy of the reacting particles is given by the electron screening potential U, when the potential becomes greater than the energy of the Gamow peak, i.e., when $U > E_{pk}$, the reaction rate depends only on the density (see Fig. 2). Cameron (1959) termed such reactions "pycnonuclear reactions" (where *pycno-* is derived from the Greek *pyknos*, "dense, compact"). Since the effective energy of the reacting particles exceeds the Gamow peak of the nonresonant reaction, *the standard strong screening enhancement factors* (Itoh *et al.* 1979; Alastuey and Jancovici 1978; Ichimaru and Utsumi 1984) *are not applicable.*

b) Application of the S-Matrix Formalism

We start with the following picture. We assume that ^{4}He, ^{8}Be, ^{12}C(0$^+$), ^{12}C*(2$^+$) and ^{12}C*(0$^+$) are eigenstates of the unperturbed Hamiltonian. When the interaction is turned on, these states undergo transitions with one another. The strength of the transitions is determined by the matrix elements of the perturbed Hamiltonian connecting these states. We can understand the meaning of an "unstable particle" within the *S*-matrix picture as follows. When the transition matrix elements of an eigenstate of the unperturbed Hamiltonian have large values for a wide range of relative energies of the incident (or scattered) particles, this eigenstate may be called an unstable particle. If we know the matrix elements, we can follow the time evolution of the eigenstates of the unperturbed Hamiltonian. Thus we can determine the abundances of ^{8}Be, ^{12}C*(0$^+$), etc. In calculating these reaction rates, we must take into account the fact that the initial state consists of identical particles. We also note that since the polarizations and angular distributions of the γ-rays emitted in the decays of the ^{12}C excited states are not observed, we can treat these decays as if they involve spin-zero particles. Below we give only the final expression obtained using the *S*-matrix formalism; further details are given in Appendix E.

For a Maxwellian distribution of initial ^{4}He nuclei, the formation rate per unit volume of the ground state of ^{12}C is given by

$$r_C = n_\alpha^3 \frac{\Gamma_{\gamma 1}}{\hbar} \left(\frac{2\pi\hbar^2}{\sqrt{M_{12} M_{23}}\, kT} \right)^3 I(E_{m1}, E_{m2}, \tilde{\Gamma}_{m1}, \Gamma_{m2}, T) \,, \tag{3.1}$$

where n_α is the number density of ^{4}He particles and M_{12} and M_{23} are the reduced masses of the ^{4}He + ^{4}He and ^{8}Be + ^{4}He systems, repectively. The function I is defined by the integral

$$I(E_{m_1}, E_{m_2}, \tilde{\Gamma}_{m_1}, \Gamma_{m_2}, T) = \frac{1}{(2\pi)^2}\int_0^\infty \int_0^\infty \frac{(1+X)^5\Gamma_{m_2}(E_2)e^{-E_2/kT}dE_2}{(E_1+E_2-E_{m_1}-E_{m_2})^2+[\Gamma_{m_2}(E_2)/2]^2}\,\frac{\tilde{\Gamma}_{m_1}(E_1)e^{-E_1/kT}dE_1}{(E_1-E_{m_1})^2+[\tilde{\Gamma}_{m_1}(E_1)/2]^2}\,, \tag{3.2}$$

where $X = (E_1 + E_2 - E_{m_1} - E_{m_2})/\Delta E(0^+ \rightarrow 2^+)$. The quantity $\Delta E(0^+ \rightarrow 2^+) = 3.2152$ MeV is the difference in energy between the ^{12}C*(0$^+$) and ^{12}C*(2$^+$) states. The quantities E_{m_1} and E_{m_2} are the resonant energies of the ^{8}Be and ^{12}C*(0$^+$) states relative to the ground states of the ^{4}He + ^{4}He and ^{8}Be + ^{4}He systems, respectively, while E_1 and E_2 are the relative kinetic energies of the ^{4}He + ^{4}He and ^{8}Be + ^{4}He systems, respectively. The quantity $\tilde{\Gamma}_{m_1}(E_1)$ is the effective energy width of the ^{8}Be ground state for decay into two ^{4}He nuclei with relative kinetic energy E_1. It is half of the standard value, $\Gamma_{m_1}(E_1)$, because the particles in the final state are identical. The quantity $\Gamma_{m_2}(E_2)$ is the energy width of the ^{12}C*(0$^+$) state for decay into ^{8}Be + ^{4}He with relative kinetic energy E_2. In order to clarify the meaning of "energy width," we make the following remarks. The quantity $\Gamma(E)$ is the product of the phase-space number density and the absolute square of the transition matrix element between the intermediate state and the particle state whose relative kinetic energy is E. It therefore represents the strength of the transition. Also, since $\Gamma(E)$ at the resonant energy gives half of the energy width appearing in the Breit-Wigner expression for the cross section, we shall call it the "energy width," even for non resonant values of E. Given this meaning, the quantity $\Gamma(E)$ is perfectly well defined for each specific energy E of the reacting particles.

When both of the energy widths are very small at their resonant energies, the integrands behave as δ-functions. We then have $I = \exp(-E_{m_1}/kT - E_{m_2}/kT)$, and we obtain the standard (Clayton 1968) thermonuclear (resonant) expression for the triple-α reaction. When the temperature is low and the density is high, we must know the form of the energy widths. For strong interactions, the energy width can be written as $\Gamma(E) = 2kR_n\gamma P(E)$, where $k = (2\mu E)^{1/2}/\hbar$ and μ is the reduced mass of the reacting particles; R_n is the radius of the nucleus; γ is the reduced energy width of the nuclear state; and $P(E)$ is a penetration factor (Bohr and Mottelson 1969). The penetration factor can be estimated using the WKB approximation. As long as the kinetic energy of the incident particle is smaller than the Coulomb energy at the surface of the nucleus ($\sim$1 MeV), the penetration factor is proportional to $E^{-1/2}\exp(-2\pi Z_1 Z_2 e^2/\hbar v)$ (for an s-wave interaction), where Z_1 and Z_2 are the charges of the colliding particles. The argument of the exponential function is derived by integrating the wavenumber in the classically forbidden region. The energy width is then simply proportional to $\exp(-bE^{-1/2})$, where $b = (2\mu)^{1/2}\pi Z_1 Z_2 e^2/\hbar$.

The electrons around a nucleus partially screen the Coulomb potential of the nucleus. The net effect of such screening in the classically forbidden region is to shift the potential by a constant value $U(<0)$ (Cameron 1959; Clayton 1968). The energy of the resonant state as determined by laboratory experiments is determined by the structure of the nucleus, in the absence of any surrounding electrons. Hence the effect of electron screening on the resonant energy is to shift it downward by an amount $|U|$. The net effect on the energy width is to move the effective energy up by an amount $|U|$. Using the index 0 to denote the values without screening, we can summarize as follows:

$$E_m = E_{m0} - |U| \qquad \text{(for the resonant energy)}\,, \tag{3.3a}$$

$$\frac{\Gamma(E)}{\Gamma_0(E)} = \exp\,[bE^{-1/2} - b(E + |U|)^{-1/2}]\,. \tag{3.3b}$$

In the pycnonuclear regime, the screening potential changes from $-|U|$ at $r = 0$ to zero at the classical turning point. Hence $\lambda|U|$ should be used in equation (3.3b) instead of $|U|$, where λ is a number between 0 and 1. The value of λ depends on the detailed structure of the screening potential. We will carry out a more detailed calculation of it in another paper. Since the final result depends only on $\lambda^{1/2}$, we shall take $\lambda = 1$ throughout the rest of this paper. In the next section, we use these relations to discuss our results.

IV. RESULTS

In order to understand the behavior of the triple-α rate, we convert the variables in the expression for it into dimensionless form. We express the energies in terms of the Gamow peak energy of the reaction, $E_{\rm pk}$. We introduce a nondimensional temperature, τ, which is related to the Gamow peak energy by $\tau = 3E_{\rm pk}/kT$. For the reaction of particles with masses M_1 and M_2 and charges Z_1 and Z_2, the parameter τ is given by

$$\tau = 3(\pi\alpha Z_1 Z_2)^{2/3}\left(\frac{Mc^2}{2kT}\right)^{1/3}\,, \tag{4.1}$$

where $M = M_1 M_2/(M_1 + M_2)$ is the reduced mass and α is the electromagnetic fine-structure constant. The quantities ϵ, c, $\gamma(\epsilon)$, and u correspond to the dimensionless energies of the relative motion of the incident particles (E), the resonant energy (E_m), the energy width [$\Gamma(E)$], and the screening potential ($|U|$), respectively. The indices 1 and 2 denote quantities and functions which are associated with the ^{4}He + ^{4}He → ^{8}Be reaction and the ^{8}Be + ^{4}He → ^{12}C*(0$^+$) reaction, respectively. The index zero refers to the values of quantities when there is no screening. With these definitions, equation (3.3) can be summarized as

$$\begin{aligned} c_1 &= c_{10} - u_1\,, & c_2 &= c_{20} - u_2\,, \\ \gamma_1(\epsilon_1) &= \gamma_{10}(\epsilon_1 + u_1)\,, & \gamma_2(\epsilon_2) &= \gamma_{20}(\epsilon_2 + u_2)\,. \end{aligned} \tag{4.2}$$

The quantity I can be written in terms of the unscreened values. When there is no screening and the reaction is purely resonant, I

is given by its classical value, $\exp(-\tau_2 c_{20}/3 - \tau_1 c_{10}/3)$. We denote this classical value by $I_{\rm cl}$. The screening factor f is defined by

$$f = \frac{I}{I_{\rm cl}} = F(c_{10}, c_{20}; \tilde{\gamma}_{10}, \gamma_{20}; \tau_1, \tau_2; u_1, u_2) , \tag{4.3}$$

where F is defined by

$$F = \frac{\gamma_{20}(c_{20})}{2\pi} \frac{\tilde{\gamma}_{10}(c_{10})}{2\pi} \exp\left[\frac{\tau_2}{3}(u_2 + c_{20} + 2c_{20}^{-1/2}) + \frac{\tau_1}{3}(u_1 + c_{10} + 2c_{20}^{-1/2})\right]$$
$$\times \int_{u_2}^{\infty} d\epsilon_2 \int_{u_1}^{\infty} d\epsilon_1 \frac{(1+X)^5 \exp[-\tau_2(\epsilon_2 + 2\epsilon_2^{-1/2})/3]}{[\epsilon_2 - c_{20} + s(\epsilon_1 - c_{10})]^2 + [\gamma_{20}(\epsilon_2)/2]^2} \frac{\exp[-\tau_1(\epsilon_1 + 2\epsilon_1^{-1/2})/3]}{(\epsilon_1 - c_{10})^2 + [\tilde{\gamma}_{10}(\epsilon_1)/2]^2} . \tag{4.4}$$

Here $X = [\epsilon_2 - c_{20} + s(\epsilon_1 - c_{10})]/\Delta\epsilon$, $\Delta\epsilon = 3\Delta E(0^+ \rightarrow 2^+)/\tau_2 kT$, and $s = \tau_1/\tau_2$. If there is no $s(\epsilon_1 - c_{10})$ term in X and in the denominator of the first integrand of equation (4.4), the integral is completely separable. Hence we call $s(\epsilon_1 - c_{10})$ a "correlation term."

The transformation from cgs units to these nondimensional units was done for two reasons. The first reason is simply that we can write the nondimensional quantity F in terms of nondimensional parameters. The second reason is that we can gain a good insight into the behavior of F without performing any numerical integrations. The parameters $\epsilon_1 = 1$ and $\epsilon_2 = 1$ correspond to the Gamow peaks of reaction 1 (^{4}He + ^{4}He → ^{8}Be) and reaction 2 [^{8}Be + ^{4}He → ^{12}C*(0$^+$)], respectively. The parameters ϵ_1 and ϵ_2 are expressed in terms of the corresponding Gamow peaks. We call them GPU1 (for "Gamow peak unit for reaction 1") and GPU2 (for "Gamow peak unit for reaction 2"). For $1 < T_6 < 10^2$, the temperatures τ_1 and τ_2 are of order 100. The function $e^{-\tau(\epsilon + 2\epsilon^{-1/2})/3}$ behaves as a δ-function of strength $(4\pi/\tau)^{1/2} e^{-\tau}$ at 1 GPU. The resonant energies c_{10} and c_{20} are of order 10 GPU1 and 10 GPU2, respectively, in this temperature range. The numerical values of the nondimensional parameters are summarized in Table 1.

Since the resonant peaks lie far above the Gamow peaks for temperatures $T \lesssim 10^8$ K, the Boltzmann factors suppress the resonant reactions and make them less important compared with the nonresonant reactions. Hence we now discuss the behavior of the nonresonant reactions. When the screening potentials u_1 and u_2 are smaller than 1 GPU1 and 1 GPU2, the integrals over ϵ_1 and ϵ_2 pick up the Gamow peaks and are independent of u_1 and u_2. The screening factor is proportional to $e^{\tau_2 u_2/3 + \tau_1 u_1/3} = e^{-U_2/kT - U_1/kT}$, which is its standard value. However, when u_1 and u_2 become larger than 1 GPU1 and 1 GPU2, the integrals do not pick up the Gamow peaks. Using the formula

$$\int_u^{\infty} e^{-\tau(\epsilon + 2\epsilon^{-1/2})/3} d\epsilon = \frac{3}{\tau}\left[1 + \frac{1}{u^{3/2}} + O\left(\frac{1}{u^{5/2}}\right)\right] e^{-\tau(u + 2u^{-1/2})/3} \quad \text{for } u > 1 , \tag{4.5}$$

we notice that the screening factor is proportional to $\exp(-2\tau_2 u_2^{-1/2}/3 - 2\tau_1 u_1^{-1/2}/3)$. If the strong screening potential is used, corresponding to a uniform electron background, we find that the argument of the exponential does not depend on temperature. It is simply proportional to $-\rho^{-1/6}$, as derived by Cameron (1959).

The physical interpretation of this behavior is as follows. The energy width increases as $\exp(-bE^{*-1/2})$ when the effective energy E^* is enhanced by screening. When the screening potential is very large, this increase is not fast enough to compensate for the decline of the Maxwellian distribution with energy. Hence, the energy of the Gamow peak decreases, and eventually it becomes negative. When this happens, only the high-energy tail of the Gamow peak contributes to the reaction rate, and *the reaction rate cannot be described by the standard nonresonant reaction rate multiplied by the standard strong screening factor* (Itoh *et al.* 1979; Alastuey and Jancovici 1978; Ichimaru and Utsumi 1984). We also note that energy widths $\tilde{\gamma}_{10}(\epsilon_1)$ and $\gamma_{20}(\epsilon_2)$ in the denominator of

TABLE 1[a]

NONDIMENSIONAL PARAMETERS IN THE TRIPLE-α REACTION

^{4}He + ^{4}He → ^{8}Be	^{8}Be + ^{4}He → ^{12}C*(0$^+$)
$\tau_1 = 134.92 T_6^{-1/3}$	$\tau_2 = 235.72 T_6^{-1/3}$
$c_{10} = 23.683 T_6^{-2/3}$	$c_{20} = 42.462 T_6^{-2/3}$
$\tilde{\gamma}_{10}(c_{10}) = 1.7547 \times 10^{-3} T_6^{-2/3}$	$\gamma_{20}(c_{20}) = 1.3145 \times 10^{-3} T_6^{-2/3}$
$u_1 = 1.3451 \rho_6^{1/3} (\mu_e/2)^{-1/3} T_6^{-2/3}$	$u_2 = 1.3535 \rho_6^{1/3} (\mu_e/2)^{-1/3} T_6^{-2/3}$
$\Delta\epsilon = 474.86 T_6^{-2/3}$	

[a] The nondimensional temperature, τ, is defined in eq. (4.1). Then the Gamow peak energy is $E_{\rm pk} = \tau kT/3$. The nondimensional energies are obtained by expressing them in terms of the Gamow peak energy. The subscripts 1 and 2 denote those quantities associated with the ^{4}He + ^{4}He → ^{8}Be and ^{8}Be + ^{4}He → ^{12}C*(0$^+$) reactions, respectively. The subscript index zero denotes the value when there is no electron screening. The quantities c and γ are the nondimensional resonance energy and resonance energy width, respectively. The quantity u is the absolute value of the nondimensional screening potential. In this paper we use the strong screening formula (5.2b). The quantity $\Delta\epsilon$ is the nondimensional resonance energy for the γ-decay, $E_{\gamma_1}[^{12}\text{C}^*(0^+) \rightarrow {}^{12}\text{C}^*(2^+)]$.

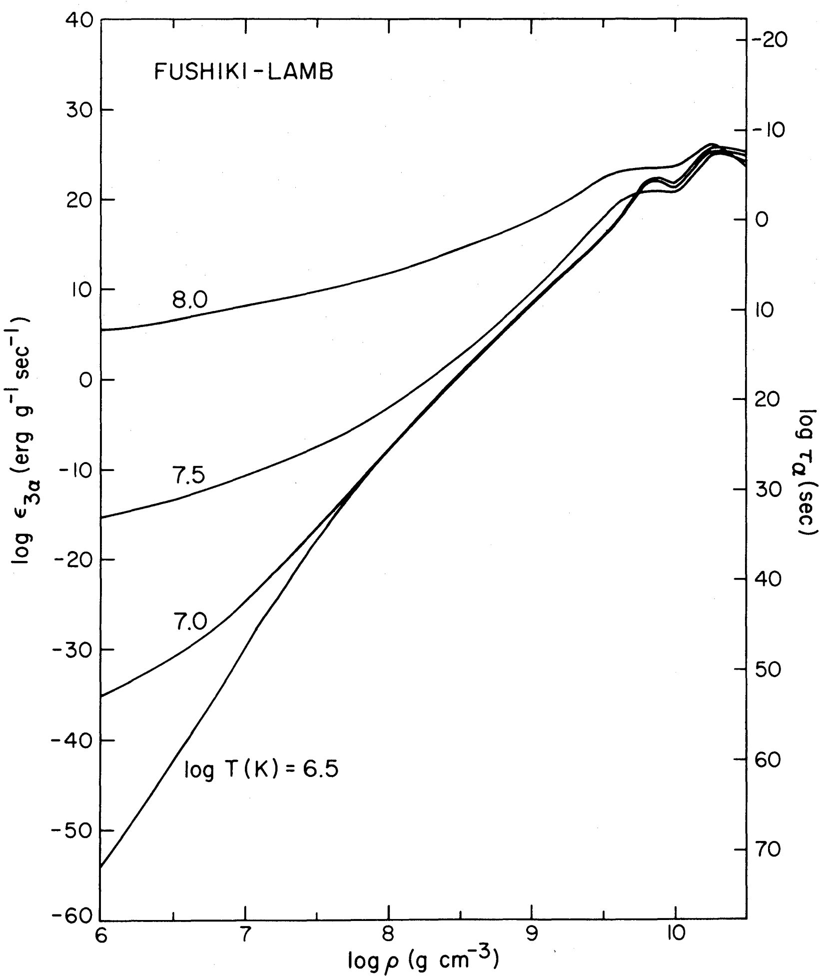

FIG. 3.—The left vertical scale gives the energy generation rate for the triple-α reaction, while the right vertical scale gives the depletion time scale for an ^{4}He particle. We assume a composition of pure ^{4}He. The curves are drawn from our numerical calculations.

equation (4.4) are important only near the resonant energies, since they do not increase fast enough to dominant the denominator (until near nuclear density).

The energy generation rate per gram is given by the product of the classical energy generation rate, $(\epsilon_{3\alpha})_{\mathrm{Cl}}$, and the factor F defined in equation (4.4):

$$\epsilon_{3\alpha} = F(\epsilon_{3\alpha})_{\mathrm{Cl}} \, ,$$

$$(\epsilon_{3\alpha})_{\mathrm{Cl}} = 5.120 \times 10^{29} Y^3 \frac{\rho_6^2}{T_6^3} e^{-4401.5/T_6} \qquad (\mathrm{ergs\ s^{-1}\ g^{-1}}) \, , \tag{4.6}$$

where T_6 and ρ_6 are the temperature and density in units of 10^6 K and 10^6 g cm^{-3}, respectively, and Y is the helium mass abundance. Figure 3 shows the energy generation rate as a function of density for fixed temperatures, based on our numerical calculations. By using an asymptotic expansion, we can write the integral (4.4) (in cgs units) as

$$\epsilon_{3\alpha} = 5.120 \times 10^{29} Y^3 \rho_6^2 G_1(T_6, \tilde{\rho}_6) G_2(T_6, \tilde{\rho}_6) \, . \tag{4.7}$$

The functions G_1 and G_2 correspond to factors which are associated with the $^4\mathrm{He} + {}^4\mathrm{He} \rightarrow {}^8\mathrm{Be}$ reaction and the $^8\mathrm{Be} + {}^4\mathrm{He} \rightarrow {}^{12}\mathrm{C}^*(0^+)$ reaction, respectively. The function G_2 includes the effect of the correlation term, about which we remarked following equation (4.4). When $u = 1.35\tilde{\rho}_6^{1/3}/T_6^{2/3} < 1$, where $\tilde{\rho}$ is defined by $\tilde{\rho} = \rho(2/\mu_e)$ (μ_e is the electron molecular weight), the functions $G_1(T_6, \tilde{\rho}_6)$ and $G_2(T_6, \tilde{\rho}_6)$ are given by

$$G_1 = \exp\left(\frac{60.492\tilde{\rho}_6^{1/3}}{T_6}\right)\left[\frac{\exp(-1065.1/T_6)}{T_6^{3/2}}\,\theta(5.458 \times 10^3 - \tilde{\rho}_6) + \frac{16.16}{T_6^{2/3}}\,\frac{\exp(-134.92/T_6^{1/3})}{(1 - 4.222 \times 10^{-2} T_6^{2/3})^2 + 2.643 \times 10^{-5} T_6^{5/3}}\right] ,$$

$$G_2 = \exp\left(\frac{106.35\tilde{\rho}_6^{1/3}}{T_6}\right)\left[\frac{\exp(-3336.4/T_6)}{T_6^{3/2}}\,\theta(1.836 \times 10^4 - \tilde{\rho}_6) + \frac{244.6}{T_6^{2/3}}\,\frac{(1 + 3.528 \times 10^{-3} T_6^{2/3})^5 \exp(-235.72/T_6^{1/3})}{(1 - 2.807 \times 10^{-2} T_6^{2/3})^2 + 2.704 \times 10^{-6} T_6^{5/3}}\right] , \tag{4.8a}$$

where $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$. When $u > 1$, they are

$$G_1 = \frac{\exp\,[(60.492\tilde{\rho}_6^{1/3} - 1065.1)/T_6]}{T_6^{3/2}}\,\theta(5.458 \times 10^3 - \tilde{\rho}_6) + \frac{1.178}{T_6^{1/2}}\left(1 + \frac{1}{u^{3/2}}\right)\frac{\exp\,(-77.554/\tilde{\rho}_6^{1/6})}{(1 - 5.680 \times 10^{-2}\tilde{\rho}_6^{1/3})^2 + 8.815 \times 10^{-7}T_6^2}\,,$$

$$G_2 = \frac{\exp\,[(106.35\tilde{\rho}_6^{1/3} - 3336.4)/T_6]}{T_6^{3/2}}\,\theta(1.836 \times 10^4 - \tilde{\rho}_6) + \frac{13.48}{T_6^{1/2}}\left(1 + \frac{1}{u^{3/2}}\right)\frac{(1 + 5.070 \times 10^{-3}\tilde{\rho}_6^{1/3})^5 \exp\,(-135.08/\tilde{\rho}_6^{1/6})}{(1 - 3.791 \times 10^{-2}\tilde{\rho}_6^{1/3})^2 + 5.162 \times 10^{-8}T_6^2}\,. \tag{4.8b}$$

These analytical expressions agree with our numerical integration to within a factor of a few for $u \sim 1$ and to within a few percent for small and large u. The contours of constant energy generation rates are shown in Figures 4 and 5.

The multiplicative factors $\exp\,(60.492\tilde{\rho}_6^{1/3}/T_6)$ in G_1 and $\exp\,(106.35\tilde{\rho}_6^{1/3}/T_6)$ in G_2 of equations (4.8a) correspond to the conventional strong screening factors (see Clayton 1968) derived from the simple screening potential (eq. [5.2b]) discussed in the next section. These factors can be replaced, if desired, by the more detailed expressions given by Itoh *et al.* (1979) for $u < 1$, and by Alastuey and Jancovici (1978) for $u < 1.6$. However, such replacements are not valid for the pycnonuclear regime described by equation (4.8b).

There are new regimes for $\log \tilde{\rho} > 9.737$ in the $^4\mathrm{He} + {}^4\mathrm{He} \rightarrow {}^8\mathrm{Be}$ reaction and for $\log \tilde{\rho} > 10.264$ in the $^8\mathrm{Be} + {}^4\mathrm{He} \rightarrow {}^{12}\mathrm{C}^*(0^+)$ reaction, respectively. Above these critical densities, the screening potentials become larger than the resonance energies of the corresponding reactions (see Fig. 2). The resonance energies therefore lie *below* the effective energy ranges for the reactions. Consequently, the reactions are exothermic and nonresonant, even at high temperature. We call these regimes "high-density nonresonant regimes"; these regimes are the reason we introduced the step functions for the resonant terms in expressions (4.8a) and (4.8b). In narrow bands in density just below the critical densities, the resonant reactions dominate even at low temperature. We call these bands "high-density resonant regimes." Since the resonant reactions turn off above the critical densities, the reaction rate drops suddenly by a factor of about 100 above the critical densities.

V. COMPARISON WITH OTHER WORK

We define three additional correction factors. The factor f_{nonres} is the ratio between the true reaction rate and the classical resonant reaction rate when electron screening is negligible. It shows how much the contribution of the nonresonant reactions enhances the reaction rate. The factor f_{pure} is the ratio between the true reaction rate and the reaction rate including the contribution of the nonresonant reactions but not the effect of electron screening. It shows how much the reaction rate is enhanced by electron screening. The factor f_{dev} shows the deviation of the screening factor from the standard formula for strong screening. Using the

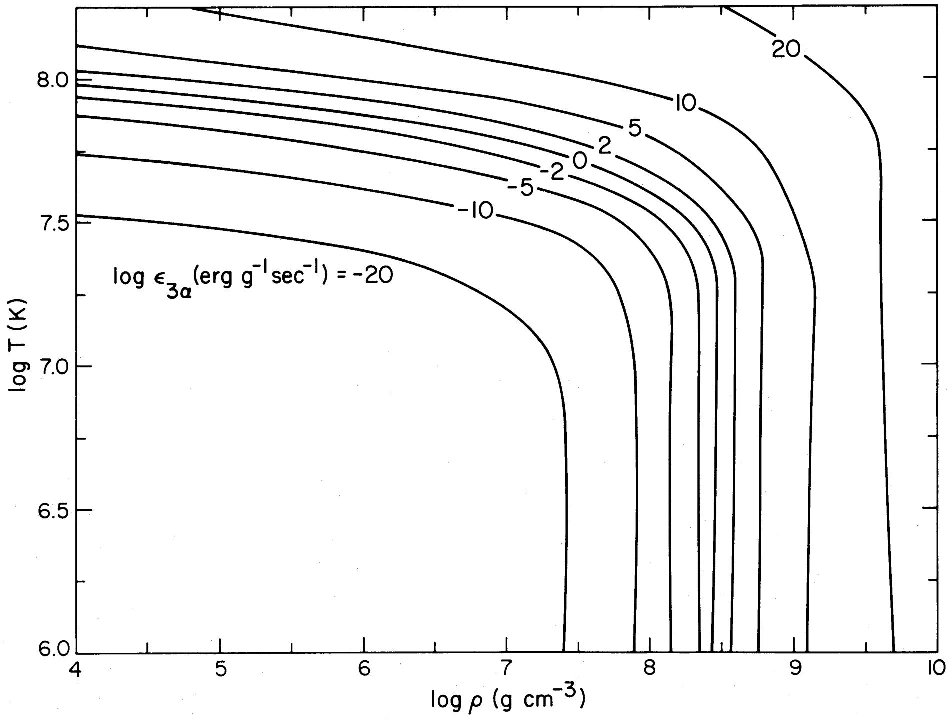

FIG. 4.—Contours of constant energy generation rate are shown in the (ρ, T)-plane

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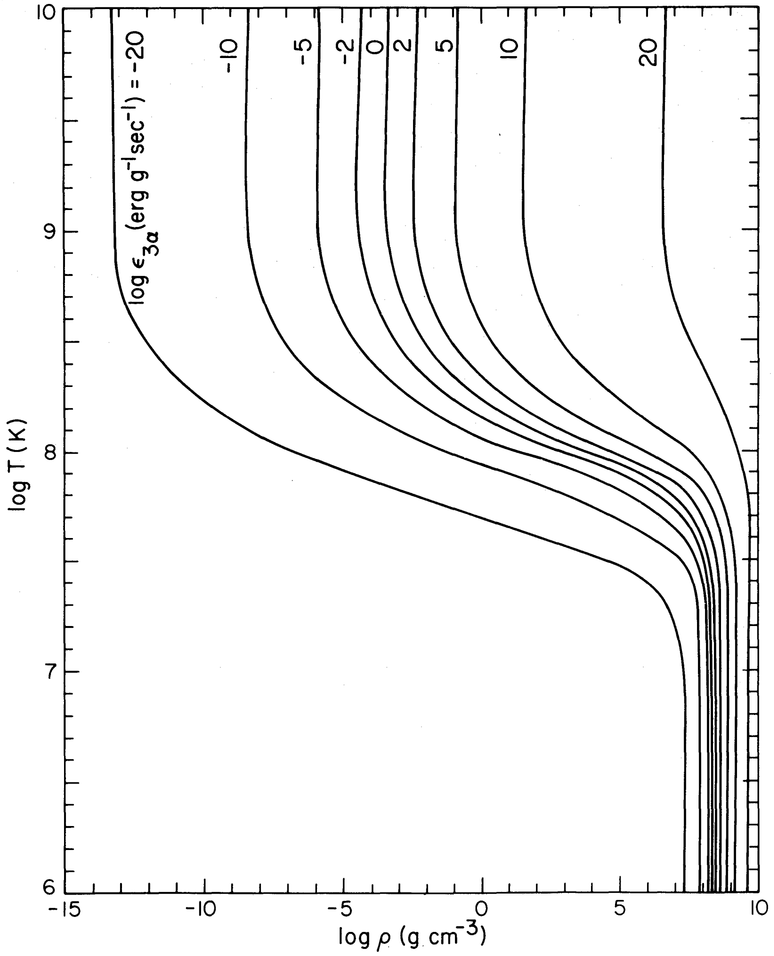

FIG. 5.—The same contours as in Fig. 4 are shown for a wider range of densities and temperatures

F-factor defined in equations (4.3) and (4.4), these three factors can be written as

$$f_{\text{nonres}} = F(c_{10}, c_{20}; \gamma_{10}, \gamma_{20}; \tau_1, \tau_2; 0, 0) ,$$

$$f_{\text{pure}} = \frac{F(c_{10}, c_{20}; \gamma_{10}, \gamma_{20}; \tau_1, \tau_2; u_1, u_2)}{F(c_{10}, c_{20}; \gamma_{10}, \gamma_{20}; \tau_1, \tau_2; 0, 0)} ,$$

$$f_{\text{dev}} = \frac{F(c_{10}, c_{20}; \gamma_{10}, \gamma_{20}; \tau_1, \tau_2; u_1, u_2)}{e^{\tau_2 u_2/3 + \tau_1 u_1/3} F(c_{10}, c_{20}; \gamma_{10}, \gamma_{20}; \tau_1, \tau_2; 0, 0)} . \tag{5.1}$$

They are shown in Figures 6*a*, 6*b*, and 6*c*. As examples, the nonresonant reaction rate is larger than the classical resonant reaction rate by a factor of 10^{500} at $\log T = 6.5$, and electron screening enhances the nonresonant reaction rate by a factor of 10^{86} at $\log T = 6.5$ and $\log \tilde{\rho} = 9.5$.

The energy generation rate calculated by Cameron (1959) is shown in Figure 7. Our reaction rate increases faster than Cameron's as the density increases. This is due to differences in the screening potentials used and in the resonant terms. Cameron used the screening potential

$$-(U)_{\text{Cameron}} = \frac{3}{2}\left(\frac{4\pi n_e}{3}\right)^{1/3} Z_1 Z_2^{2/3} e^2 , \tag{5.2a}$$

while we use

$$-(U)_{\text{Fushiki-Lamb}} = \frac{9}{10}\left(\frac{4\pi n_e}{3}\right)^{1/3} [(Z_1 + Z_2)^{5/3} - Z_1^{5/3} - Z_2^{5/3}] e^2 . \tag{5.2b}$$

As a consequence, Cameron's ^{4}He + ^{4}He → ^{8}Be reaction rate increases as $\exp(-65.1/\tilde{\rho}_6^{1/6})$, while ours increases as $\exp(-77.6/\tilde{\rho}_6^{1/6})$. The ^{8}Be + ^{4}He → ^{12}C*(0$^+$) reaction rate shows the same tendency. Cameron did not include electron screening in the resonant terms, while we have done so (see expressions [4.8a] and [4.8b]). The reaction rates also differ for two additional reasons. First,

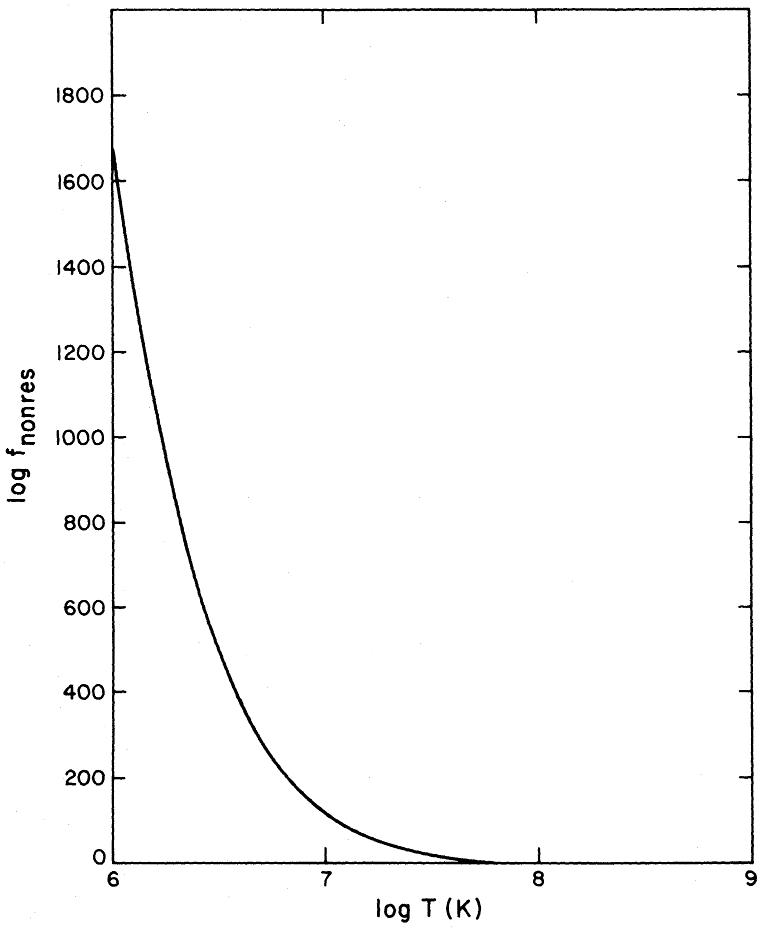

FIG. 6*a*

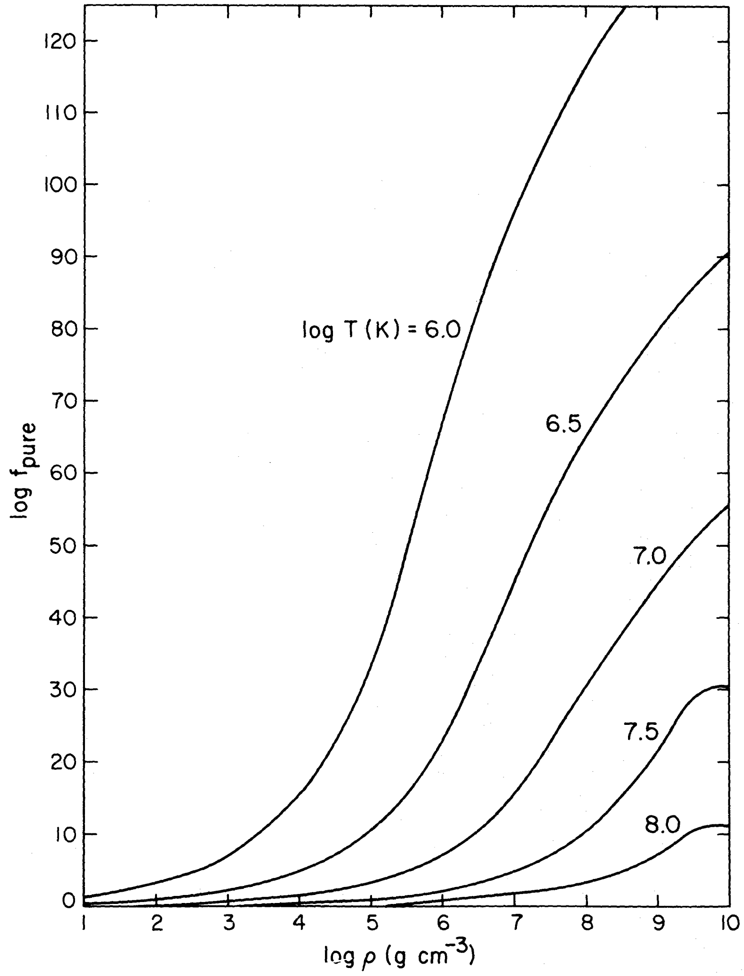

FIG. 6*b*

FIG. 6.—(*a*) The factor $f_{\rm nonres}$ defined in eq. (5.1) is shown as a function of T. (*b*) The factor $f_{\rm pure}$ defined in eq. (5.1) is shown as a function of density for fixed temperatures. (*c*) The factor $f_{\rm dev}$ defined in eq. (5.1) is shown for different temperatures.

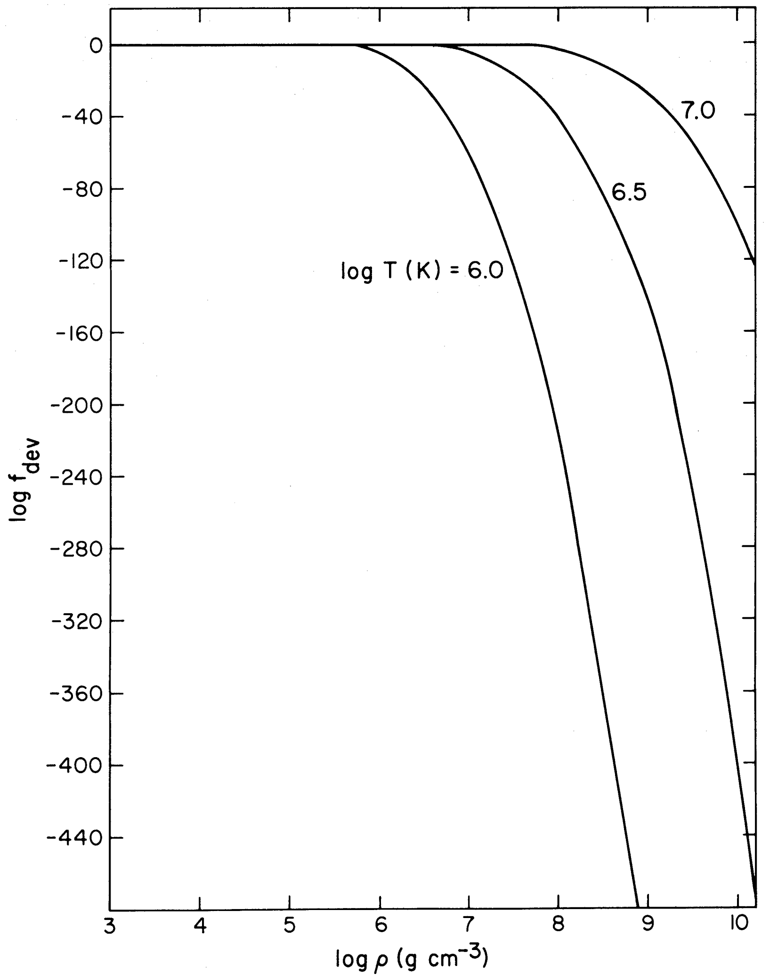

FIG. 6c

Cameron used 1.4 eV for the width of the $^{12}C^*(0^+)$ to $^{12}C^*(2^+)$ radiative transition, while we use the more recent experimental value of 3.7×10^{-3} eV (see Table 2). Second, Cameron dropped the $E^{-1/2}$ dependence in the penetration factor. As a consequence, the energy widths of α-capture processes acquired $E^{1/2}$ dependence. When averaged over the initial energies, this leads to a factor $kT/(E_{m_1} E_{m_2})^{1/2}$ compared with our calculation. Both of these factors are of order 10^3 in the strong screening regime; however, they go in opposite directions and the overall error is therefore much smaller.

The energy generation rate calculated by NTM is shown in Figure 8. When the correlation term in the reaction rate is omitted [see our remark following eq. (4.4)] and the density is low ($u < 1$), our results agree with those of NTM. However, when the correlation term is included, we find that the energy generation rate is reduced only by a factor $[E_{m_2}/(E_{m_1} + E_{m_2})]^2 \sim 0.6$, whereas NTM's energy generation rate decreases to less than a few hundredths of its former value. Furthermore, NTM give an expression for the unscreened reaction rate only. In the weak and strong electron screening regimes, the screened reaction rate may be found simply by multiplying their expression by the standard screening factor (e.g., Itoh *et al.* 1979). However, as we discussed earlier, this approach cannot be used in the pycnonuclear regime. What would happen were one to do so is illustrated by the extensions of

TABLE 2

COMPARISON OF EXPERIMENTAL PARAMETERS USED BY VARIOUS AUTHORS

Parameter	Fushiki-Lamb	Cameron	NTM
E_{m_1}	91.78 keV	94 keV	91.78 keV
E_{m_2}	287.5 keV	278 keV	287.7 keV
E_{γ_1}	3.2152 MeV	...	3.2175 MeV
E_{γ_2}	4.439 MeV	...	...
Γ_{m_1}	6.8 eV	...	6.8 eV
Γ_{m_2}	8.9 eV	...	8.5 eV
Γ_{γ_1}	3.7 meV	1.4 eV	3.7 meV
Γ_{γ_2}	10.8 meV	...	...

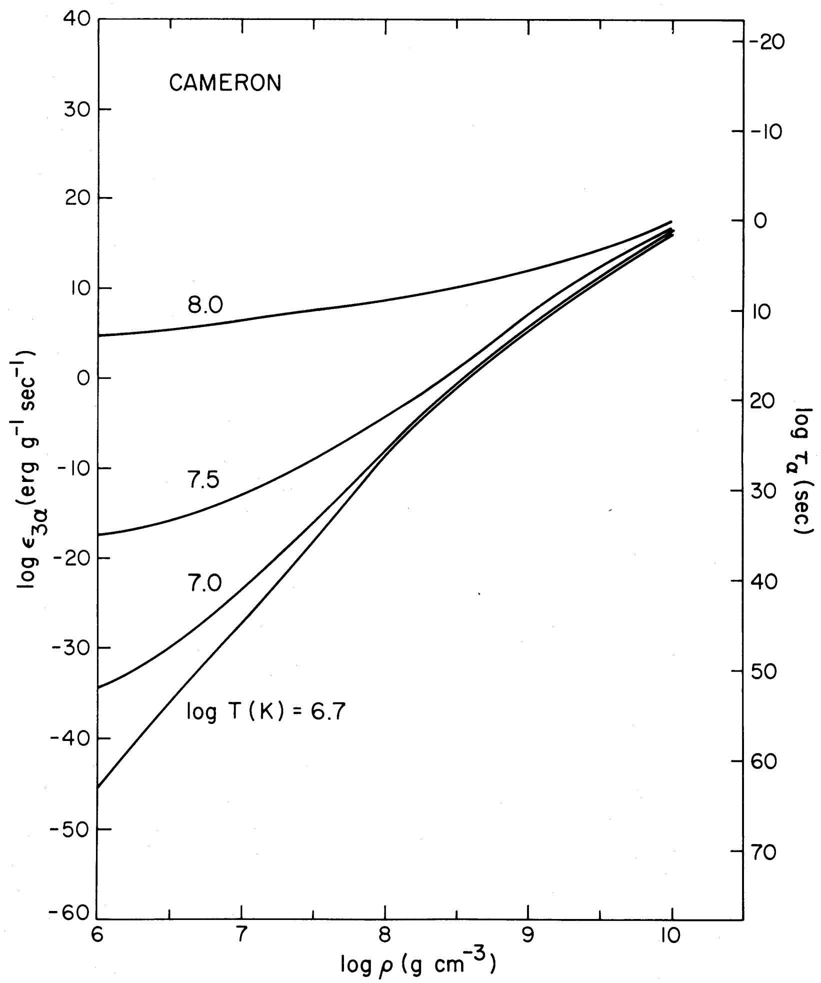

FIG. 7

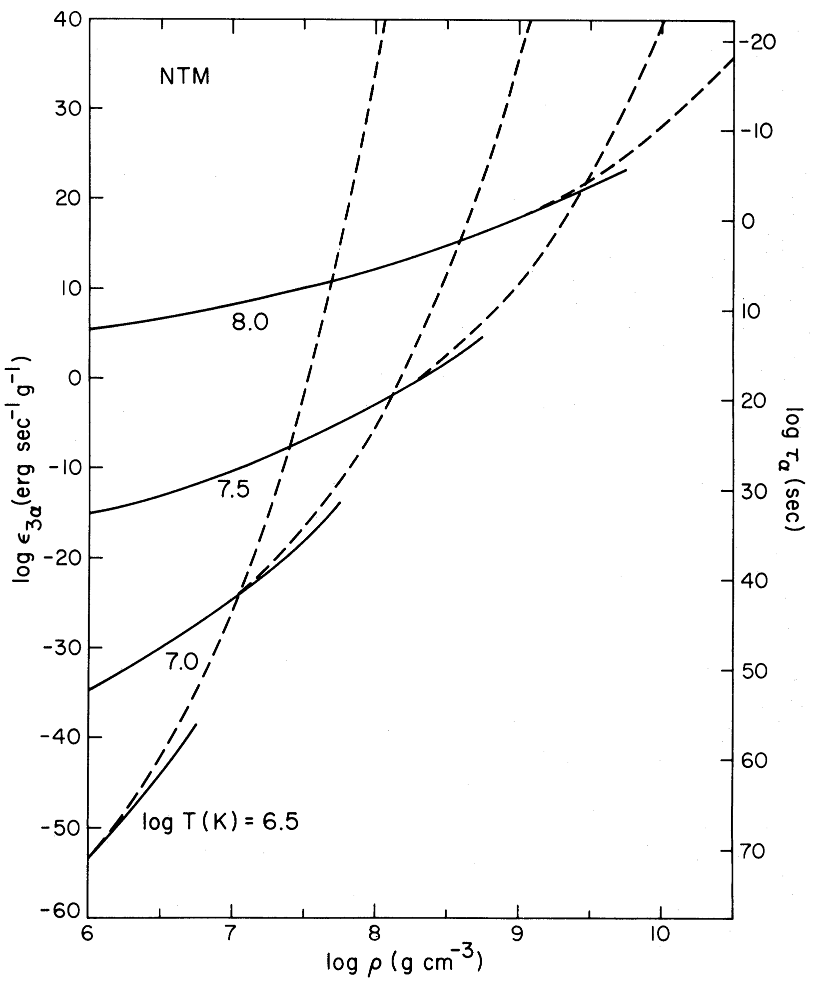

FIG. 8

FIG. 7.—Same as Fig. 3, except calculated by us from the results of Cameron (1959)

FIG. 8.—Same as Fig. 3, except calculated by us from the results of Nomoto, Thielemann, and Miyaji (1985)

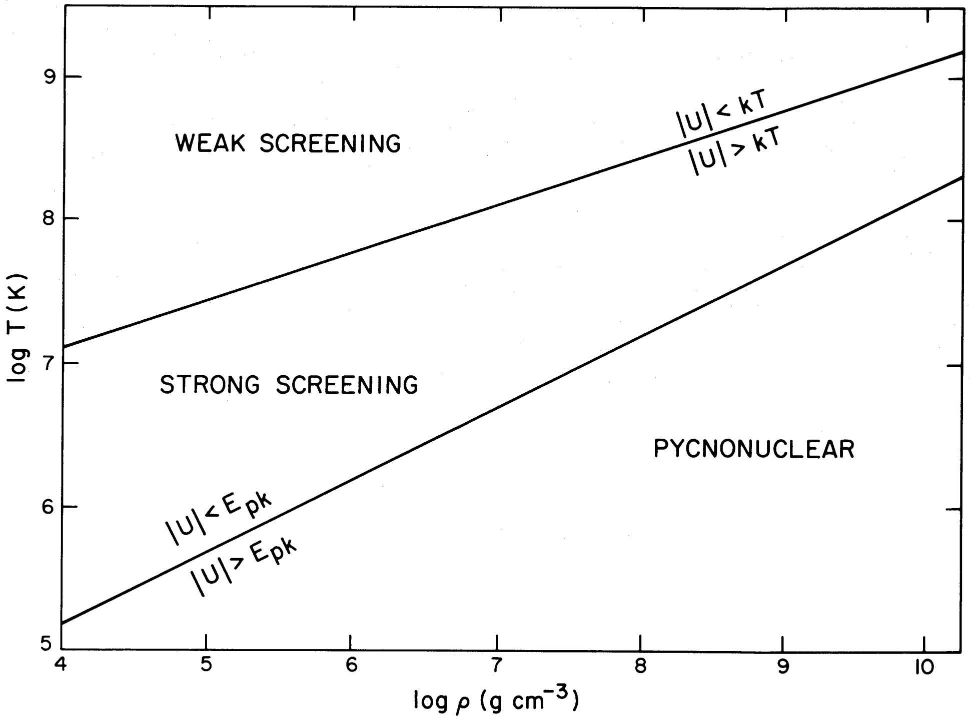

FIG. 9.—Weak, strong, and pycnonuclear screening regimes for the ^{4}He + ^{4}He → ^{8}Be reaction, assuming a pure helium composition. The quantities $|U|$ and $E_{\rm pk}$ are the screening potential and the Gamow peak energy for the reaction.

NTM's curves shown as dashed lines in Figure 8. We calculated these lines by multiplying NTM's unscreened reaction rate by the strong screening factor obtained using our simple expression for the screening potential ($f = e^{|U|/kT}$). Thus NTM's expression for the reaction rate can be used in the weak and strong screening regimes, but not in the pycnonuclear regime. Our analytical expression (eqs. [4.7] and [4.8]) is valid in all three regimes. Figure 9 shows these regimes for the reaction ^{4}He + ^{4}He → ^{8}Be [the regimes for the reaction ^{8}Be + ^{4}He → ^{12}C(0^+) are very similar]. Table 2 summarizes the experimental parameters used by Cameron, NTM, and ourselves.

VI. CONCLUSIONS

We have developed an *S*-matrix formalism which can be applied to reactions in which electron screening is important, including three-body reactions and reactions involving weak interactions. The *S*-matrix approach allows us to define precisely the meaning of the energy-dependent width of the reaction in terms of the interaction Hamiltonian. The reaction rate can then be calculated quantum mechanically from first principles or by using the experimental value of the energy width at resonance, as we have done in this paper.

We have discussed systematically the various regimes of the triple-α reaction, and have identified a new nonresonant regime at high ($\log \rho > 9.7$) densities. Using the *S*-matrix formalism, we have obtained an analytical expression for the screened triple-α reaction which is accurate for all temperatures and densities. We have compared our results with those of Cameron (1959) and NTM, and have verifed NTM's expression for the unscreened reaction rate. However, we have shown that the reaction rate in the pycnonuclear regime cannot be obtained from the unscreened rate using a screening factor, and that NTM's expression therefore cannot be used in this regime. We confirm the $e^{-C/\rho^{1/6}}$ dependence of the pycnonuclear reaction rate (Cameron 1959).

We are grateful to Fred Lamb for advice about the *S*-matrix formalism and its application to three-body reactions, and to Friedel Thielemann for discussions about the triple-α reaction and for sharing his calculations with us. We thank Chris Pethick and George Victor for helpful comments about scattering theory, and Geoff Ravenhall for helping us to locate the most up-to-date experimental values for the relevant nuclear reactions. We also thank Al Cameron, Willy Fowler, Dai-ichiro Sugimoto, and Jim Truran for stimulating discussions about electron screening. One of us (I. F.) gratefully acknowledges the hospitality of the Department of Astronomy and Astrophysics and the Enrico Fermi Institute at the University of Chicago, where this work was completed. This research was supported in part by National Aeronautics and Space Administration grants NAG-8520, NAG-8563, NAGW-246, and NAGW-830.

APPENDIX A

INTERACTION PICTURE

Suppose that there are $(k + 1)$ stable states when there is no interaction between them. When the interaction is turned on, these states are no longer stable and they undergo transitions with one another. Let us assume that the total Hamiltonian can be separated into two terms,

$$H = H_0 + H_{\text{int}} , \tag{A1}$$

where H_0 and H_{int} denote the unperturbed and interaction parts, respectively. Let us further assume that the eigenstates $|n\rangle$ $(n = 0, 1, 2, \ldots, k)$ of H_0 are complete and that both H_0 and H_{int} are Hermitian operators. The eigenstates satisfy

$$H_0|n\rangle = E_n|n\rangle , \qquad \langle n|m\rangle = \mathcal{N}_m \delta_{nm} , \tag{A2}$$

where $\mathcal{N}_m$ are positive normalization constants. Since the unperturbed Hamiltonian is Hermitian, the unperturbed energies E_n are real numbers. An arbitrary state $|\alpha(t)\rangle$ of the perturbed system can be expanded in terms of unperturbed states with time-dependent numbers $b_n(t)$ as

$$|\alpha(t)\rangle = \sum_n b_n(t) e^{-iE_nt/\hbar}|n\rangle .$$

Using the interaction picture, the Schrödinger equation can be written as

$$i\hbar \frac{\partial}{\partial t} |\alpha'(t)\rangle = H'_{\text{int}}|\alpha'(t)\rangle$$

or

$$i\hbar \mathcal{N}_n^{1/2} \dot{b}_n(t) = \sum_m H_{nm} e^{i(E_n - E_m)t/\hbar} \mathcal{N}_m^{1/2} b_m(t) , \tag{A3a}$$

where

$$\begin{aligned} |\alpha'(t)\rangle &= e^{iH_0t/\hbar}|\alpha(t)\rangle , \\ H'_{\text{int}} &= e^{iH_0t/\hbar} H_{\text{int}} e^{-iH_0t/\hbar} , \\ H_{nm} &= \langle n|H_{\text{int}}|m\rangle / (\mathcal{N}_n \mathcal{N}_m)^{1/2} . \end{aligned} \tag{A3b}$$

If we choose the state $|\alpha(t)\rangle$ that coincides with the state $|0\rangle$ at $t = 0$ (or equivalently $b_n(0) = \delta_{n0}$), $\mathcal{N}_n b_n(t)$ gives the S-matrix element $\langle n|S(t; 0)|0\rangle$. Thus the S-matrix gives the abundance of each state as a function of the time t.

In ordinary perturbation theory, $b_n(t)$ is expanded in terms of the order of $\hbar$ and the first few terms are calculated. This approximation is valid only as long as the amplitude of a state $b_n(t)$ does not decay significantly or oscillate. However, we must be able to handle just such a situation in order to calculate the abundances of the states. We must therefore find a nonperturbative approach.

Following the procedure in Heitler (1954), we use outgoing Green's functions for the $b_n(t)$:

$$b_n(t) = -\frac{1}{2\pi i} \int_C dE \; G_n(E) e^{i(E_n - E)t/\hbar} . \tag{A4}$$

The contour C of integration is shown in Figure 10. The $G_n(E)$ satisfy

$$(E - E_n)\mathcal{N}_n^{1/2} G_n(E) = \sum_m H_{nm} \mathcal{N}_m^{1/2} G_m(E) + \mathcal{N}_0^{1/2} \delta_{n0} . \tag{A5}$$

Our problem is to solve equation (A5) in a nonperturbative way. It can be easily proved that all poles of $G_n(E)$ are real as long as we consider only discrete states. However, when we allow for continuous states, the Green's functions have imaginary poles. The outgoing Green's functions of the continuous states can be defined by analytic continuation of the Green's functions for the discrete states from the upper plane of E. By using the formula

$$\frac{1}{z - a} = \mathrm{P}\frac{1}{z - a} - i\pi\delta(z - a) , \tag{A6}$$

where a is a real number, we can obtain the outgoing Green's functions for the continuous states.

APPENDIX B

A RESONANT REACTION

Here we study the simplest resonant reaction. Let us suppose that the states $|0\rangle, |1\rangle, |2\rangle, \ldots, |k\rangle$, and $|m\rangle$ are orthogonal eigenstates of the unperturbed Hamiltonian. When the interaction is turned on, these states undergo transitions with each other. We

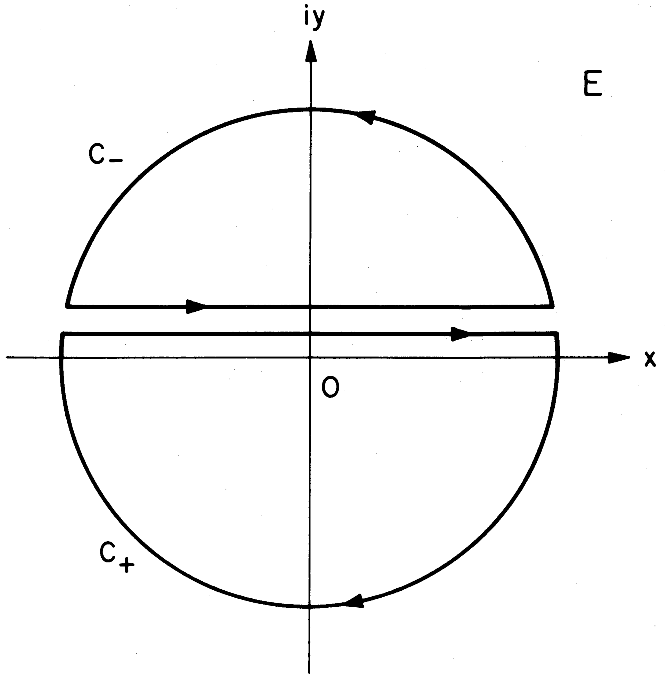

FIG. 10.—Contour taken in the E-plane to give an outgoing wave function. The contour C_+ is taken for $t > 0$, and the contour C_- taken for $t < 0$.

regard the state $|m\rangle$ as the intermediate state, according to the following meaning: the other states cannot undergo transitions with one another unless they pass through the state $|m\rangle$.

Taking the state $|0\rangle$ as the initial state, we can follow the time evolution of all of the states; we call this the "$1 + 1 + k$ states problem." The resulting Green's functions can be written in the form (A5) as

$$
\begin{aligned}
(E - E_0)\mathcal{N}_0^{1/2} G_0(E) &= H_{0m} G_m(E) + \mathcal{N}_0^{1/2} , \\
(E - E_m) G_m(E) &= H_{m0} \mathcal{N}_0^{1/2} G_0(E) + \sum_{j=1}^{k} H_{mj} G_j(E) , \\
(E - E_j) G_j(E) &= H_{jm} G_m(E) ,
\end{aligned}
\tag{B1}
$$

where we take the normalization of the states as $\mathcal{N}_m = \mathcal{N}_j = 1$ $(j = 1, 2, \ldots, k)$. The solution is

$$
\begin{aligned}
G_0(E) &= \left\{(E - E_0) - \frac{|H_{m0}|^2}{E - E_m - \sum_{j=1}^{k}[|H_{jk}|^2/(E - E_j)]}\right\}^{-1} , \\
G_m(E) &= \frac{\mathcal{N}_0^{1/2} H_{m0}}{(E - E_0)\{E - E_m - \sum_{j=1}^{k}[|H_{jm}|^2/(E - E_j)]\} - |H_{m0}|^2} , \\
G_j(E) &= \frac{\mathcal{N}_0^{1/2} H_{jm} H_{m0}}{(E - E_j)[\![(E - E_0)\{E - E_m - \sum_{j'=1}^{k}[|H_{j'm}|^2/(E - E_j')]\} - |H_{m0}|^2]\!]} .
\end{aligned}
\tag{B2}
$$

Regarding the states $|0\rangle, |1\rangle, \ldots, |k\rangle$ as continuous states and using the result of Appendix C, we can write the outgoing Green's functions as

$$
\begin{aligned}
G_0^+(E) &= \left[(E - E_0) - \frac{|H_{m0}|^2}{E - E_m - i\Gamma(E)/2}\right]^{-1} \\
G_m^+(E) &= \frac{\mathcal{N}_0^{1/2} H_{m0}}{(E - E_0)[E - E_m - i\Gamma(E)/2] - |H_{m0}|^2} , \\
G_j^+(E) &= \frac{\mathcal{N}_0^{1/2} H_{jm} H_{m0}}{(E - E_j)\{(E - E_0)[E - E_m - i\Gamma(E)/2] - |H_{m0}|^2\}} .
\end{aligned}
\tag{B3}
$$

where the energy shifts are included in E_0, E_m, and E_j.

The energy width $\Gamma(E)$ is defined by

$$
\Gamma(E) = 2\pi |H_{jm}|^2 \rho(E_j)|_{E_j = E} ,
\tag{B4}
$$

where $\rho(E_j)$ is the number of states per unit energy interval of all the possible final channels. If we neglect the $|H_{m0}|^2$ term, $G_0^+(E)$ is simply $1/(E - E_0)$, and the initial state remains unchanged. Hence, we expect that we can neglect the $|H_{m0}|^2$ terms in the denominators of the outgoing Green's functions as long as the abundance of the initial state does not change appreciably. We discuss the validity of this approximation in Appendix D.

The contour integrals of the outgoing Green's functions give the amplitudes of the states, and the values of the poles of the Green's functions are therefore important. We evaluate them at the resonant energies and regard the energy widths as constants. We obtain

$$|b_m(t)|^2 = \frac{\mathcal{N}_0|H_{m0}|^2}{(E_0 - E_m)^2 + (\Gamma/2)^2}\left\{1 - 2e^{-\Gamma t/2\hbar}\cos\left[\frac{(E_0 - E_m)t}{\hbar}\right] + e^{-\Gamma t/\hbar}\right\}. \tag{B5}$$

This expression has a rather unfamiliar form. However, if we average the initial energy E_0 over a small interval around E_m, we obtain the familiar form

$$\langle|b_m(t)|^2\rangle = \frac{\mathcal{N}_0|H_{m0}|^2(1 - e^{-\Gamma t/\hbar})}{(E_0 - E_m)^2 + (\Gamma/2)^2}. \tag{B6}$$

When $\Gamma t/\hbar \gg 1$, the state $|m\rangle$ approaches its equilibrium value. The energy width Γ is of order 1 eV for the strong interaction and 1 meV for the electromagnetic interaction. The time scales for reactions involving these interactions to reach equilibrium are therefore typically 10^{-15} and 10^{-12} s, respectively. Both conditions are well satisfied in the case of interest to us. Under these circumstances, the production rate of the state $|j\rangle$ is

$$\begin{aligned}\frac{d}{dt}|b_j(t)|^2 \approx \frac{2}{\hbar}\frac{\mathcal{N}_0|H_{jm}|^2|H_{m0}|^2}{(E_0 - E_m)^2 + (\Gamma/2)^2}&\left\{\frac{1}{E_j - E_0}\sin\left(\frac{E_j - E_0}{\hbar}t\right)\right.\\ &\left.- \frac{1}{(E_j - E_m)^2 + (\Gamma/2)^2}\left[(E_j - E_m)\sin\left(\frac{E_j - E_0}{\hbar}t\right) + \frac{\Gamma}{2}\cos\left(\frac{E_j - E_0}{\hbar}t\right)\right]\right\}.\end{aligned} \tag{B7}$$

Let us suppose that there are q exit channels, and denote one of them as Q. The creation rate for the Q particle is given by integrating over its momentum as

$$\frac{d}{dt}|b_Q(t)|^2 = \int_{j \in Q}\frac{d}{dt}|b_j(t)|^2\rho_Q(E_j)dE_j \approx \frac{2\pi}{\hbar}\frac{\mathcal{N}_0\,\rho_Q(E_j)|H_{jm}|^2|H_{m0}|^2}{(E_0 - E_m)^2 + (\Gamma/2)^2}\bigg|_{E_j = E_0}, \tag{B8}$$

where $\rho_Q(E_j)$ denotes the number density of the Q channel. Since integration over the second and third terms in equation (B7) gives $e^{-\Gamma t/2\hbar}$, these terms drop out and we obtain a constant creation rate. Such a constant creation rate cannot be true at arbitrarily long times, because the absolute square of $b_Q(t)$ is limited. We come back to this problem in Appendox D.

APPENDIX C

GREEN'S FUNCTIONS FOR CONTINUOUS STATES

The poles of the Green's functions for discrete states are real as long as the Hamiltonian is Hermitian. However, when the number of real poles is increased, the Green's functions cannot be continuous at the real axis. Therefore, we cannot obtain the Green's functions for continuous states simply by increasing the number of poles in the Green's functions for discrete states. We must define outgoing and ingoing Green's functions for continuous states which are analytic across the real axis.

Let us consider the complex function $F(z)$ defined by

$$F(z) = \int_\alpha^\beta \frac{da}{z - a}, \tag{C1}$$

where α and β are real numbers and the integral is performed along the real axis. The discrete form of $F(z)$ is

$$F_n(z) = \sum_{k=0}^{n}\frac{1}{z - a_k}, \tag{C2}$$

where $\alpha < a_0 < a_1 < \cdots < a_n = \beta$. Apparently there are no imaginary poles in $F_n(z)$. By "imaginary pole," we mean a pole which has an imaginary part; it need not be purely imaginary. As n increases, the strip on the real axis between α and β becomes filled with poles. The function $F_n(z)$ is ill-behaved on this strip and is not extensible across the strip. We now show that in the continuum limit, the strip should be treated as a cut rather than as a collection of real poles.

We define two different analytic functions, depending on from which plane we analytically extend $F(z)$. The integral in $F(z)$ is then easily evaluated. We obtain

$$F(z) = [-\ln(a - z)]_\alpha^\beta = -\ln\left(\frac{\beta - z}{\alpha - z}\right). \tag{C3}$$

Notice that there is no singularity in the strip $\alpha < z < \beta$. We restrict our attention to the region where $\alpha < \text{Re}\, z < \beta$. We define the outgoing function $F^+(z)$ by extending $F(z)$ from the upper plane to the lower plane of z. When $\text{Im}\, z > 0$, the argument of $a - z$ increases by an amount $\phi^+(z) = \arg(\alpha - z) - \arg(\beta - z)$, where $\pi < \arg(\alpha - z) < 3\pi/2 < \arg(\beta - z) < 2\pi$. Using the function $\phi^+(z)$, we can write $F^+(z)$ as

$$F^+(z) = -\ln\left|\frac{\beta - z}{\alpha - z}\right| - i\phi^+(z)\,. \tag{C4}$$

If $\text{Im}\, z$ is small, $\phi^+(z) \sim \pi$. By choosing $\pi/2 < \arg(\alpha - z) \leq \pi$ and $2\pi \leq \arg(\beta - z) < 5\pi/2$, we can extend $F^+(z)$ to the lower half-plane. The function $\phi^+(z)$ is still $\sim\pi$ when the imaginary part of z is small. We define the ingoing function $F^-(z)$ by choosing $-\pi/2 < \arg(\beta - z) < \pi/2$ and $\pi/2 < \arg(\alpha - z) < \pi$. We define $\phi^-(z)$ as the imaginary part of $-F^-(z)$. This definition is similar to that for $\phi^+(z)$ in terms of $F^+(z)$. Contrary to the case for $\phi^+(z)$, the function $\phi^-(z) \sim -\pi$ when the imaginary part of z is small.

When the integral is of the form

$$F_1(z) = \int_\alpha^\beta \frac{f(a)da}{z - a}\,, \tag{C5}$$

where $f(z)$ does not have a pole, we can define the outgoing and ingoing functions as follows. If the imaginary part of z is small, we can choose α_1 and β_1 ($\alpha < \alpha_1 < \text{Re}\, z < \beta_1 < \beta$), so that $f(z)$ does not change appreciably in the neighborhood of α_1 and β_1. Hence we can write

$$F_1(z) \approx \int_\alpha^{\alpha_1} \frac{f(a)da}{z - a} + f(z)\int_{\alpha_1}^{\beta_1} \frac{da}{z - a} + \int_{\beta_1}^{\beta} \frac{f(a)da}{z - a}\,. \tag{C6}$$

Choosing $|\alpha_1 - z| = |\beta_1 - z|$, we obtain the outgoing and ingoing functions

$$F_1^\pm(x) = \mathsf{P}\int_\alpha^\beta \frac{f(a)da}{z - a} \mp i\pi f(x)\,. \tag{C7}$$

In the integral following P, the argument of z is chosen so that $\arg(z - \alpha_1) \sim \arg(\beta_1 - z)$. When z is real, P reduces to the standard principal-value integral.

Symbolically, we can write

$$\frac{1}{z - a} = \mathsf{P}\,\frac{1}{z - a} \mp i\pi\delta(z - a)\,, \tag{C8}$$

where the minus sign is for the outgoing function and the plus sign is for the ingoing function.

APPENDIX D

EXACT SOLUTION FOR A RESONANT REACTION

In Appendix B we derived the amplitudes of the states by dropping $|H_{m0}|^2$ in the denominators of the Green's functions. In this appendix we derive the exact solutions and justify the approximation made in Appendix B.

First, we start with the outgoing Green's function $G_0^+(E)$ for the initial state. It can be expanded in the form

$$G_0^+(E) = \frac{1}{E - E_0}\sum_{n=0}^{\infty}\left[\frac{|H_{m0}|^2}{(E - E_0)(E - E_m + i\Gamma/2)}\right]^n\,. \tag{D1}$$

The amplitude $b_0(t)$ is easily obtained from the Cauchy integral:

$$\begin{aligned} b_0(t) &= -\frac{1}{2\pi i}\int_C G_0^+(E)e^{i(E - E_0)t/\hbar}dE \\ &= \sum_{r=0}^{\infty}\sum_{n=r}^{\infty}\frac{(n + r - 1)!}{r!(n - 1)!}\,\frac{1}{(n - r)!}\left[\frac{-i|H_{m0}|^2 t}{\hbar(E_0 - E_m + i\Gamma/2)}\right]^{n-r}\left[-\frac{|H_{m0}|^2}{(E_0 - E_m + i\Gamma/2)^2}\right]^r \\ &\quad + e^{i(E_0 - E_m)t/\hbar - \Gamma t/2\hbar}\sum_{r=0}^{\infty}\sum_{n=r}^{\infty}\frac{(n + r + 1)!}{r!(n + 1)!}\,\frac{1}{(n - r)!}\left[\frac{i|H_{m0}|^2 t}{\hbar(E_0 - E_m + i\Gamma/2)}\right]^{n-r}\left[-\frac{|H_{m0}|^2}{(E_0 - E_m + i\Gamma/2)^2}\right]^{r+1}\,. \end{aligned} \tag{D2}$$

For $t \gg \hbar/\Gamma$, the second series of the expansion drops out, and we have

$$|b_0(t)|^2 = \exp\left[-\frac{\Gamma|H_{m0}|^2}{(E_0 - E_m)^2 + (\Gamma/2)^2}\,\frac{t}{\hbar}\right]\left[1 + O\left(\frac{|H_{m0}|^2}{(E_0 - E_m)^2 + (\Gamma/2)^2}\right)\right]\,. \tag{D3}$$

Similarly, the amplitude $b_m(t)$ can be derived by expanding the outgoing Green's function in terms of $|H_{m0}|^2$ and using the Cauchy integral. For $t \gg \hbar/F$, we obtain

$$b_m(t) = \frac{\mathcal{N}_0^{1/2} H_{m0}}{E_0 - E_m + i\Gamma/2} \exp\left(-i \frac{|H_{m0}|^2}{E_0 - E_m + i\Gamma/2} \frac{t}{\hbar}\right)\left[1 + O\left(\frac{|H_{m0}|^2}{(E_0 - E_m + i\Gamma/2)^2}\right)\right] . \tag{D4}$$

The absolute square magnitude of the amplitude is

$$|b_m(t)|^2 = \frac{\mathcal{N}_0 |H_{m0}|^2}{(E_0 - E_m)^2 + (\Gamma/2)^2} \exp\left[-\frac{\Gamma |H_{m0}|^2}{(E_m - E_0)^2 + (\Gamma/2)^2} \frac{t}{\hbar}\right]\left[1 + O\left(\frac{|H_{m0}|^2}{(E_0 - E_m)^2 + (\Gamma/2)^2}\right)\right] . \tag{D5}$$

Integrating $i\hbar \dot{b}_j(t) = H_{jm} e^{i(E_j - E_m)t/\hbar} b_m(t)$, we can obtain the amplitudes $b_j(t)$. The final abundance of the channel Q, $|b_Q(t)|^2$, is given by summing over the j which belongs to the channel. For $t \gg \hbar[(E_0 - E_m)^2 + (\Gamma/2)^2]/\Gamma |H_{m0}|^2$, the equilibrium abundance of the final state is

$$|b_Q(t)|^2 = \mathcal{N}_0 \frac{\Gamma_Q}{\Gamma} . \tag{D6}$$

In § II, we chose $\mathcal{N}_0 = N_1 N_2$. However, in order to have the solution consistent to all orders, we notice that $\mathcal{N}_0$ should be chosen to be N_2, where we assume that $N_2 \leq N_1$. The reason is as follows. The sum of the absolute squares of the amplitudes is a constant value $\mathcal{N}_0$. This value is also equal to the maximum possible number of the intermediate state. Since the intermediate state is formed from particles 1 and 2, the maximum possible number should be N_2—that is, the number of the less abundant particle. The decay time of the initial and intermediate states should be equal to the creation time of the final state. In order to have a consistent picture, the initial wave function of particle 1 should be normalized to be unity within a volume $1/n_1$, where n_1 is the number density of particle 1, instead of being normalized to N_1 in the volume V. With this normalization, $|H_{m0}|^2$ can be written as

$$|H_{m0}|^2 = \pi \hbar \lambdabar^2 n_1 v \Gamma_0 , \tag{D7}$$

and the decay time scale τ for the initial state becomes

$$\frac{1}{\tau} = \pi \lambdabar^2 \frac{n_1 v \Gamma \Gamma_0}{(E_m - E_0)^2 + (\Gamma/2)^2} . \tag{D8}$$

This coincides with the formation rate for the final state for a specific relative energy E_0 per particle 2. Thus, as long as t is less than τ, our approximation in Appendix B and § II is valid. As far as this condition is satisfied, the results in Appendix B and § II are independent of the normalization. We may choose $\mathcal{N}_0 = N_1 N_2$.

APPENDIX E

S-MATRIX FOR THE TRIPLE-ALPHA REACTION

In this appendix we formulate the S-matrix for the triple-α reaction. We define $|0\rangle$ as the initial state, which is made of three ^{4}He particles. First, we assume that these three particles are distinguishable. The ^{4}He particles 1 and 2 form a ^{8}Be nucleus whose momentum is the same as the total momentum of particles 1 and 2. The $|m_1\rangle$ corresponds to the system consisting of the formed ^{8}Be nucleus and the third ^{4}He particle. This state $|m_1\rangle$ can decay into a series of states $|\sigma_1\rangle$. In our case, the $|\sigma_1\rangle$ correspond to the system consisting of the third ^{4}He particle and the two ^{4}He particles from the decay of ^{8}Be. The state $|m_1\rangle$ can also form $|m_2\rangle$, which consists of a ^{12}C*(0$^+$) nucleus whose momentum is the same as the total momentum of the state $|m_1\rangle$. The state $|m_2\rangle$ can decay into a series of states $|\sigma_2\rangle$ and $|\mu_1\rangle$. The states $|\sigma_2\rangle$ correspond to the system consisting of the decayed ^{8}Be nucleus and a ^{4}He nucleus. The states $|\mu_1\rangle$ correspond to the system consisting of the emitted γ-ray and the ^{12}C*(2$^+$) nucleus. The states $|\mu_1\rangle$ can decay into the states $|\mu_2\rangle$, which correspond to the system consisting of the ^{12}C(0$^+$) nucleus and two emitted γ-rays. We take $\mathcal{N}_0$ for the normalization constant of the initial state, and the other states are normalized to unity. We can then write the basic equations for the Green's functions as

$$\begin{aligned}
(E - E_0)\mathcal{N}_0^{1/2} G_0(E) &= H_{0m} G_m(E) + \mathcal{N}_0^{1/2} , \\
(E - E_{m_1}) G_{m_1}(E) &= H_{m0} \mathcal{N}_0^{1/2} G_0(E) + H_{m_1 m_2} G_{m_2}(E) + \sum_{\sigma_1} H_{m_1 \sigma_1} G_{\sigma_1}(E) , \\
(E - E_{\sigma_1}) G_{\sigma_1}(E) &= H_{\sigma_1 m_1} G_{m_1}(E) , \\
(E - E_{m_2}) G_{m_2}(E) &= H_{m_2 m_1} G_{m_1}(E) + \sum_{\sigma_2} H_{m_1 \sigma_2} G_{\sigma_2}(E) + \sum_{\mu_1} H_{m_2 \mu_1} G_{\mu_1}(E) , \\
(E - E_{\sigma_2}) G_{\sigma_2}(E) &= H_{\sigma_2 m_2} G_{m_2}(E) , \\
(E - E_{\mu_1}) G_{\mu_1}(E) &= H_{\mu_1 m_2} G_{m_2}(E) + \sum_{\mu_2} H_{\mu_1 \mu_2} G_{\mu_2}(E) , \\
(E - E_{\mu_2}) G_{\mu_2}(E) &= H_{\mu_2 \mu_1} G_{\mu_1}(E) .
\end{aligned} \tag{E1}$$

The solutions for ^{12}C* and the final states are

$$G_{\mu_2}(E) = \frac{H_{\mu_2\mu_1}H_{\mu_1 m_2}}{(E - E_{\mu_2})\{E - E_{\mu_1} - \sum_{\mu'_2}[|H_{\mu_1\mu'_2}|^2/(E - E_{\mu'_2})]\}} G_{m_2}(E) , \tag{E2}$$

$$\begin{aligned} \frac{\mathscr{N}_0^{1/2} H_{m_2 m_1} H_{m_1 0}}{G_{m_2}(E)} &= \left\{E - E_{m_2} - \sum_{\sigma_2} \frac{|H_{m_2\sigma_2}|^2}{E - E_{\sigma_2}} - \sum_{\mu_1} \frac{|H_{m_2\mu_1}|^2}{E - E_{\mu_1} - \sum_{\mu_2}[|H_{\mu_1\mu_2}|^2/(E - E_{\mu_2})]}\right\}\left(E - E_{m_1} - \sum_{\sigma_1} \frac{|H_{m_1\sigma_1}|^2}{E - E_{\sigma_1}}\right)(E - E_0) \\ &\quad - |H_{0m_1}|^2\left\{E - E_{m_2} - \sum_{\sigma_2} \frac{|H_{m_2\sigma_2}|^2}{E - E_{\sigma_2}} - \sum_{\mu_1} \frac{|H_{m_2\mu_1}|^2}{E - E_{\mu_1} - \sum_{\mu_2}[|H_{\mu_1\mu_2}|^2/(E - E_{\mu_2})]}\right\} - |H_{m_1 m_2}|^2(E - E_0) . \end{aligned} \tag{E3}$$

As long as the initial state does not change appreciably, we can neglect the $|H_{m_1 0}|$ and $|H_{m_2 m_1}|$ terms in the denominators of the Green's functions. Since

$$\lim_{\mathrm{Im}\, E \to +0} \sum_{\mu_1} \frac{|H_{m_2\mu_1}|^2}{E - E_{\mu_1} - \sum_{\mu_2}[|H_{\mu_1\mu_2}|^2/(E - E_{\mu_2})]} = \lim_{\mathrm{Im}\, E \to 0} \sum_{\mu_1} \frac{|H_{m_2\mu_1}|^2}{E - E_{\mu_1} - \mathrm{P}\sum_{\mu_2}[|H_{\mu_1\mu_2}|^2/(E - E_{\mu_2})] + i\Gamma_{\mu_1}(E)/2} , \tag{E4}$$

where $\Gamma_{\mu_1}(E)$ is the total decay width of the state $|\mu_1\rangle$. The energy width in this equation is given by $\Gamma_{m_2\mu_1}$, as long as $\Gamma_{\mu_1}(E)$ is small compared with the resonant energy E_{μ_1}. The energy width $\Gamma_{m_2\mu_1}$ is the decay width of the state $|m_2\rangle$ into the states $|\mu_1\rangle$. We then obtain for the outgoing function of the state $|m_2\rangle$ the expression

$$G^+_{m_2}(E) = \frac{\mathscr{N}_0^{1/2} H_{m_2 m_1} H_{m_1 0}}{[E - E_{m_2} + i\Gamma_{m_2}(E)/2][E - E_{m_1} + i\tilde{\Gamma}_{m_1}(E)/2](E - E_0)} . \tag{E5}$$

The energy shifts are included in E_{m_1} and E_{m_2}. The width $\Gamma_{m_2}(E)$ is the total decay width of the state $|m_2\rangle$; i.e., it is the sum of the decay width $[\Gamma_{m_2\sigma_2}(E)]$ to the ^{8}Be + ^{4}He system and the decay width $[\Gamma_{m_2\mu_1}(E)]$ to the ^{12}C*(2$^+$) + γ system. Since the latter width is a thousand times smaller, the total width is equal to $\Gamma_{m_2\sigma_2}(E)$. The effective width $\tilde{\Gamma}_{m_1}(E)$ is used because a ^{8}Be nucleus decays into two identical particles. The amplitude $b_{m_2}(t)$ is given by the Green's function. When $t \gg \hbar/\Gamma_{m_1}(E_{m_1}), \hbar/\Gamma_{m_2}(E_{m_2})$, we obtain for the equilibrium abundance the expression

$$|b_{m_2}(t)|^2 \approx \frac{\mathscr{N}_0 |H_{m_2 m_1}|^2 |H_{m_1 0}|^2}{[(E_0 - E_{m_1})^2 + (\Gamma_{m_1}/2)^2][(E_0 - E_{m_2})^2 + (\tilde{\Gamma}_{m_1}/2)^2]} . \tag{E6}$$

Here we do not specify the arguments of the energy widths. We will come back to this point later. The outgoing Green's function for the final state is

$$G^+_{\mu_2}(E) = \frac{\mathscr{N}_0^{1/2} H_{\mu_2\mu_1} H_{\mu_1 m_2} H_{n_2 m_1} H_{m_1 0}}{(E - E_{\mu_2})(E - E_{\mu_1} + i\Gamma_{\mu_1}/2)(E - E_{m_2} + i\Gamma_{m_2}/2)(E - E_{m_1} + i\tilde{\Gamma}_{m_1}/2)(E - E_0)} . \tag{E7}$$

The amplitude $b_{\mu_2}(t)$ is derived from this Green's function. It has a rather complicated form even for $t \gg \hbar/\Gamma_{m_1}(E_{m_1}), \hbar/\Gamma_{m_2}(E_{m_2}), \hbar/\Gamma_{\mu_1}(E_{\mu_1})$. After integrating over the energies of the emitted photons, we obtain a constant reaction rate

$$\frac{d}{dt} |b_{^{12}\mathrm{C}}|^2 = \frac{\Gamma_{\gamma_1}(E_{\gamma_1})|_{E_{\mu 1} = E_0}}{\hbar} \frac{\mathscr{N}_0 |H_{m_2 m_1}|^2 |H_{m_1 0}|^2}{[(E_0 - E_{m_2})^2 + (\Gamma_{m_2}/2)^2][(E_0 - E_{m_1})^2 + (\tilde{\Gamma}_{m_1}/2)^2]} , \tag{E8}$$

where $\Gamma_{\gamma_1}(E_{\gamma_1}) = \Gamma_{m_2\mu_2}$. In this derivation we have assumed that the decay widths of the γ-ray transitions are narrow compared with the resonance energies and that they do not have an exponential dependence on the energies.

In order to obtain quantities which are physically observable, we must average over the initial states. The initial energy E_0 is the sum of the kinetic energy of particles 1, 2, and 3. We can decompose it as

$$E_0 = E_1 + E_2 + E_3 = E_{12} + E_{1+2} + E_3 = E_{12} + E_{(12)3} + E_{\mathrm{CM}} . \tag{E9}$$

The quantities E_{12} and E_{1+2} are the relative energy and the center-of-mass energy of the 1 + 2 system, respectively. The quantities $E_{(12)3}$ and E_{CM} are the relative energy and the center-of-mass energy of the (1 + 2) + 3 system, respectively. We denote by $\tilde{E}_{m_1}$ and $\tilde{E}_{m_2}$ the threshold energies of the ^{4}He + ^{4}He → ^{8}Be and ^{8}Be + ^{4}He → ^{12}C*(0$^+$) reactions, respectively. The energies E_{m_1} and E_{m_2} can be written as

$$E_{m_1} = \tilde{E}_{m_1} + E_{1+2} + E_3 , \qquad E_{m_2} = \tilde{E}_{m_1} + \tilde{E}_{m_2} + E_{\mathrm{CM}} . \tag{E10}$$

We assume that $|H_{m_1 0}|$ and $|H_{m_2 m_1}|$ are functions, respectively, of E_{12} and $E_{(12)3}$ only. Also, we assume that the energy distribution function of the initial state $f_0(E_0)$ can be decomposed into the product of the distribution functions of E_{12}, $E_{(12)3}$, and E_{CM}, i.e.,

$$f_0(E_0) = f_{12}(E_{12}) f_{(12)3}(E_{(12)3}) f_{\mathrm{CM}}(E_{\mathrm{CM}}) .$$

Since we have assumed that the matrix elements of the interaction Hamiltonian are independent of E_{CM}, the integral of $f_{\mathrm{CM}}(E_{\mathrm{CM}})$ over E_{CM} gives unity, and the integrations reduce to the double integral over E_{12} and $E_{(12)3}$. From now on we shall use E_0 for $E_0 - E_{\mathrm{CM}}$, E_1 for E_{12}, E_2 for $E_{(12)3}$, $f_1(E_1)$ for $f_{12}(E_{12})$, $f_2(E_2)$ for $f_{(12)3}(E_{(12)3})$, E_{m_1} for $\tilde{E}_{m_1}$, and E_{m_2} for $\tilde{E}_{m_2}$. In this new notation, E_{μ_1} is equal to $E_{\gamma_1} + E_{m_1} + E_{m_2} - \Delta E$, where ΔE is the energy gap between ^{12}C*(0$^+$) and ^{12}C*(2$^+$). Setting $E_{\mu_1} = E_0$ gives

$$E_{\gamma_1} = \Delta E + E_1 + E_2 - E_{m_1} - E_{m_2} . \tag{E11}$$

Since $^{12}C^*(0^+) \rightarrow {}^{12}C^*(2_+)$ is an E2 transition, its energy width, $\Gamma_{\gamma 1}(E_{\gamma 1})$, can be written as

$$\Gamma_{\gamma 1}(E_{\gamma 1})\Big|_{E_{\mu 1}=E_0} = \Gamma_{\gamma 1} \times \left(1 + \frac{E_1 + E_2 - E_{m_1} - E_{m_2}}{\Delta E}\right)^5 , \tag{E12}$$

where $\Gamma_{\gamma 1}$ is the energy width at the resonant energy.

So far we have not considered the indistinguishability of three ^{4}He particles. We now do so. When the interaction Hamiltonian operates on the state $|\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3\rangle$, where $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, and $\boldsymbol{p}_3$ are the momenta of the three α-particles, there are three different ways to form $^{12}C^*(0^+)$. If we assume that the products between different paths vanish, we have three times the contribution to the absolute square values of the amplitudes. However, since the particles are indistinguishable, the total phase space is 3! times smaller than that of three distinguishable particles. The factor by which we must multiply the absolute square values of the amplitudes is therefore $3/3! = \frac{1}{2}$. We take $\mathcal{N}_0 = N^3$, where N is the total number of ^{4}He particles in the volume V. The average number density of $^{12}C^*(0^+)$ nuclei is given by the average of $|b_{m_2}(t)|^2/V$. We can write this as

$$n_{^{12}C^*(0^+)} = n_\alpha^3 \left(\frac{2\pi\hbar^2}{\sqrt{M_{12} M_{23}}}\right)^3 K_1(E_{m_1}, E_{m_2}, \tilde{\Gamma}_{m_1}, \Gamma_{m_2}) , \tag{E13}$$

where the function K_1 is defined by

$$K_1(E_{m_1}, E_{m_2}, \tilde{\Gamma}_{m_1}, \Gamma_{m_2}) = \frac{1}{16\pi} \int_0^\infty \int_0^\infty \frac{\Gamma_{m_2}(E_2) E_2^{-1/2} f_2(E_2) dE_2}{(E_1 + E_2 - E_{m_1} - E_{m_2})^2 + [\Gamma_{m_2}(E_2)/2]^2} \frac{\tilde{\Gamma}_{m_1}(E_1) E_1^{-1/2} f_1(E_1) dE_1}{(E_1 - E_{m_1})^2 + [\tilde{\Gamma}_{m_1}(E_1)/2]^2} . \tag{E14}$$

The average reaction rate per volume ($\{r_C\}$) is given by the average of $(d|b_{^{12}C}|^2/dt)/V$. This can be written as

$$\{r_C\} = n_\alpha^3 \frac{\Gamma_{\gamma 1}}{\hbar} \left(\frac{2\pi\hbar^2}{\sqrt{M_{12} M_{23}}}\right)^3 K_2(E_{m_1}, E_{m_2}, \tilde{\Gamma}_{m_1}, \Gamma_{m_2}) , \tag{E15}$$

where the function K_2 is defined by

$$K_2(E_{m_1}, E_{m_2}, \tilde{\Gamma}_{m_1}, \Gamma_{m_2}) = \frac{1}{16\pi} \int_0^\infty \int_0^\infty \frac{(1 + X)^5 \Gamma_{m_2}(E_2) E_2^{-1/2} f_2(E_2) dE_2}{(E_1 + E_2 - E_{m_1} - E_{m_2})^2 + [\Gamma_{m_2}(E_2)/2]^2} \frac{\tilde{\Gamma}_{m_1}(E_1) E_1^{-1/2} f_1(E_1) dE_1}{(E_1 - E_{m_1})^2 + [\tilde{\Gamma}_{m_1}(E_1)/2]^2} , \tag{E16}$$

and $X = (E_1 + E_2 - E_{m_1} - E_{m_2})/\Delta E$. When the distribution functions are Maxwellian, we obtain the results given in § IV.

If the typical energies of E_1 and E_2 are close to E_{m_1} and E_{m_2}, respectively, the average reaction rate can be written as

$$\{r_C\} \approx \frac{\Gamma_{\gamma 1}}{\hbar} n_{^{12}C^*(0^+)} . \tag{E17}$$

REFERENCES

Ajzenberg-Selove, F. 1984, *Nucl. Phys. A*, **413**, 1.

Ajzenberg-Selove, F., and Busch, C. L. 1980, *Nucl. Phys. A*, **336**, 1.

Alastuey, A., and Jancovici, B. 1978, *Ap. J.*, **226**, 1034.

Bohr, A., and Mottelson, B. R. 1969, *Nuclear Structure*, Vol. **1** (New York: Benjamin), p. 358.

Cameron, A. G. W. 1959, *Ap. J.*, **130**, 916.

Clayton, D. D. 1968, *Principles of Stellar Evolution and Nuclear Synthesis* (New York: McGraw-Hill).

Cook, C., Fowler, W. A., Lauritsen, C. C., and Lauritsen, T. 1957, *Phys. Rev.*, **107**, 508.

Fowler, W. A., Caughlan, G. R., and Zimmerman, B. A. 1967, *Ann. Rev. Astr. Ap.*, **5**, 525.

Fowler, W. A., and Greenstein, J. L. 1956, *Proc. Nat. Acad. Sci*, **42**, 173.

Hameury, J. M., Bonazzola, S., Heyvaerts, J., and Lasota, J. P. 1984, *Adv. Space Res.*, No. 10–12, p. 297.

Heitler, W. 1954, *The Quantum Theory of Radiation* (3d ed.; Oxford: Clarendon).

Hoyle, F. 1954, *Ap. J. Suppl.*, **1**, 121.

Ichimaru, S., and Utsumi, K. *Ap. J.*, **278**, 382.

Itoh, N., Totsuji, H., Ichimaru, S., and DeWitt, H. E. 1979, *Ap. J.*, **234**, 1079; **239**, 414.

Lamb, F. K., and ter Haar, D. 1971, *Phys. Rept.*, **2**, 253.

Lewin, W. H. G., and Joss, P. C. 1983, in *Accretion Driven Stellar X-Ray Sources*, ed. W. H. G. Lewin and E. P. J. van den Heuvel (Cambridge: Cambridge University Press), p. 115.

Miyaji, S., and Nomoto, K. 1985, *Astr. Ap.*, **152**, 33.

Nomoto, K. 1982, *Ap. J.*, **253**, 798.

Nomoto, K., Thielemann, F. K., and Miyaji, S. 1985, *Astr. Ap.*, **149**, 239 (NTM).

Salpeter, E. E. 1952, *Ap. J.*, **115**, 326.

———. 1953, *Ann. Rev. Nucl. Sci.*, **2**, 41.

———. 1957, *Phys. Rev.*, **107**, 516.

Salpeter, E. E., and Van Horn, H. M. 1969, *Ap. J.*, **155**, 183.

Taam, R. E. 1985, *Ann. Rev. Nucl. Particle Sci.*, **35**, 1.

Woosley, S. E. 1984, in *AIP Conf. Proc. No. 115, High Energy Transients in Astrophysics*, ed. S. E. Woosley (New York: AIP), p. 597.

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