Cosmology Notes: 10/12/06

*** Start reading Chapter 3 of Dodelson. ***

Let's take a look at the proper distance versus redshift at the *current* epoch. Recall that along a radial trajectory, i.e. $\theta = \phi = 0$, we have

$$\begin{aligned} d\vec{l}^2 &= R^2(t)\frac{dr^2}{1-kr^2} \\ \Rightarrow dl &= R(t)\frac{dr}{\sqrt{1-kr^2}}. \end{aligned} \tag{1}$$

So we have that the proper distance l is just the integral of (1), from some initial from $l = 0$ to the current l:

$$l = \int_0^l dl' = \int_0^{r_0} R(t_0)\frac{dr}{\sqrt{1-kr^2}} = R_0 \int_0^{r_0} \frac{dr}{\sqrt{1-kr^2}}. \tag{2}$$

But, you will recall from our previous lectures that we called this integral $\chi(z)$, or in other words:

$$l = R_0\chi(z) = \frac{c}{H_0}\int_{1/(1+z)}^{1} \frac{dx}{x\sqrt{\frac{\Omega_0}{x}+1-\Omega_0}}. \tag{3}$$

Now we consider what is the relation between cosmic time and redshift in a matter only universe. We start from the Friedmann equation:

$$\dot{R}^2 = R_0^2 H_0^2 \left(\Omega_0 \frac{R_0}{R} + 1 - \Omega_0\right)$$

$$\Rightarrow dt = \frac{dR}{R_0 H_0 \sqrt{\Omega_0 \frac{R_0}{R} + 1 - \Omega_0}}.$$

Now, t is found by integrating from 0 to t the left hand side, while integrating from 0 to $R(t)$ on the right hand side. If we let

$$x = \frac{R}{R_0} \qquad \Rightarrow \qquad dx = \frac{dR}{R_0}$$

then we have

$$t = \frac{1}{H_0}\int_0^{1/(1+z)} \frac{dx}{\sqrt{\frac{\Omega_0}{x}+1-\Omega_0}}, \tag{4}$$

where we have used the fact that $\frac{R(t)}{R_0} = \frac{1}{1+z}$.

So, lets take an example: the Einstein-deSitter universe, where $\Omega_0 = 1$. Plugging this into (4) we have trivially

$$t = \frac{1}{H_0}\int_0^{1/(1+z)} \sqrt{x}dx = \frac{2}{3}\frac{1}{H_0}\left(\frac{1}{1+z}\right)^{3/2}. \tag{5}$$

Notice that in the limits

$$\lim_{z\to\infty} t = 0, \qquad \lim_{z\to 0} t = \frac{2}{3}\frac{1}{H_0}.$$

It has been worked out[1], for an for arbitrary Friedmann models with $\Omega_\Lambda = 0$:

$$\begin{aligned}
\Omega_0 > 1:\ t &= \frac{\Omega_0}{2H_0(\Omega_0-1)^{3/2}}\left[\cos^{-1}\left(\frac{\Omega_0 z - \Omega_0 + 2}{\Omega_0(1+z)}\right) - \frac{2\sqrt{(\Omega_0-1)(\Omega_0 z+1)}}{\Omega_0(1+z)}\right] \\
\Omega_0 < 1:\ t &= \frac{\Omega_0}{2H_0(1-\Omega_0)^{3/2}}\left[\frac{2\sqrt{(1-\Omega_0)(\Omega_0 z+1)}}{\Omega_0(1+z)} - \cosh^{-1}\left(\frac{\Omega_0 z - \Omega_0 + 2}{\Omega_0(1+z)}\right)\right] \\
\Omega_0 = 0:\ t &= \frac{1}{H_0}\frac{1}{1+z}.
\end{aligned}$$

Let us now consider the proper distance taking into account the expansion of the universe along the trajectory. In other words, we want the proper distance versus redshift for *any* epoch:

$$ds^2 = dt^2 - \frac{d\vec{l}^2}{c} = 0 \quad \Rightarrow \quad cdt = -R(t)\frac{dr}{\sqrt{1-kr^2}} = -dl,$$

where we have used (1) and introduced a "$-$" sign such that as t gets earlier and earlier, l gets longer and longer. So from this we can say that over any finite distance we have

$$c\Delta t = -\Delta l.$$

Now we investigate the relation between the proper distance and the comoving distance. Consider two objects which are separated by some distance d_0 today. We know from our previous discussions about scaling that this distance at any time t can be written as

$$d_0 R_0 = dR(t) \quad \Rightarrow \quad d = \frac{d_0 R_0}{R(t)} = \frac{d_0}{1+z}$$

[1] Although, it is probably not useful today, but **whatever**.

and therefore

$$d_0 = d(1+z). \tag{6}$$

The left hand side of (6) is called the comoving distance and the d in the right hand side is the proper distance. Similarly, we can consider the comoving and proper *volume*:

$$V_0 = V(1+z)^3.$$

Note that the comoving volume (i.e. the volume which a volume V at redshift z would occupy today) is always larger than the proper volume: the universe is expanding.

Let us examine the proper volume in more detail. Imagine you are an observer looking at the night sky through some solid angle Ω. If we know your solid angle and the distance to the object, we can get the area, A, of the sky you are looking at. Now we consider some *volume* of the sky you are examining which has (along your line of sight) one side at the distance which corresponds to the area A, and the far side at this distance plus some length Δl. Now, if Δl is small, then we can say that the areas at either location are equal; i.e. we are looking at a volume of a rectangular box:

$$V = A\Delta l = Ac\Delta t.$$

We know that the area covered by some solid angle is related to the angular diameter distance (similar to how θ is related to that distance):

$$\theta = \frac{\text{Length}}{d_{ang}};\ \Omega = \frac{\text{Area}}{d_{ang}^2} \quad \Rightarrow \quad A = \Omega d_{ang}^2,$$

and therefore

$$V = \Omega d_{ang}^2 c\Delta t. \tag{7}$$

We return to our example of the Einstein-deSitter universe whose time was given by (5). If we differentiate this expression and turn the differentials into finite differences we have

$$|\Delta t| = \frac{1}{H_0}(1+z)^{-5/2}\Delta z.$$

We now need to recall our expression for the angular diameter distance:

$$d_{ang} = \frac{d_{lum}}{(1+z)^2}; \qquad d_{lum} = \frac{2c}{H_0}\left((1+z) - \sqrt{1+z}\right).$$

Plugging this back into (7) we see that

$$
\begin{aligned}
V &= \frac{4c^2\Omega}{H_0^2}\left(\frac{1}{1+z}-\frac{1}{(1+z)^{3/2}}\right)^2 c\left[\frac{1}{H_0}\frac{1}{(1+z)^{5/2}}\Delta z\right] \\
&= \frac{4c^3\Omega\Delta z}{H_0^3}\left(\frac{1}{(1+z)^{9/2}}-\frac{2}{(1+z)^5}+\frac{1}{(1+z)^{11/2}}\right)
\end{aligned} \tag{8}
$$

Now we turn our attention to something called *surface brightness*. When we look at a star through a telescope, it seems as a pointsource; i.e. we can't resolve the image but rather we measure a total energy flux from the source. When looking at galaxies, we can resolve them because they are distributed objects. We take the flux we obtain per a pixel, and integrate over all pixels to get the total brightness, in our direction, of the object. This is called the surface brightness S. We therefore have

$$
\begin{aligned}
S &= \text{energy}\cdot(\text{time})^{-1}\cdot(\text{area})^{-1}\cdot(\text{solid angle})^{-1} \\
S_\lambda &= \text{“} \qquad\qquad\qquad\qquad\qquad\qquad \text{”}\cdot(\text{ wavelength})^{-1}.
\end{aligned} \tag{9}
$$

We can now consider some solid angle $d\Omega$ which makes an angle θ with respect to the normal of some surface area element dA, and we write the differential amount of energy seen from this source as:

$$dE = S_\lambda dt\ dA\ d\Omega\cos\theta\ d\lambda.$$

Therefore, we can write the total surface flux from the source as

$$F = \int S d\Omega.$$

In Euclidean geometry, we know that $d\Omega = \frac{dA}{r^2}$ and also that the flux is proportial to r^{-2}, therefore the surface brightness is independent of distance. If we write $S = \frac{F}{\Omega}$ and recall the definitions of F and Ω:

$$F = \frac{L}{4\pi d_{lum}^2};\quad \Omega = \frac{A}{d_{ang}^2} = \frac{A}{d_{lum}^2(1+z)^{-4}}$$

we can see that the surface brightness is

$$S = \frac{L}{4\pi A(1+z)^4} = S_0(1+z)^{-4}.$$

Similarly, if we express the flux as per unit wavelength/frequency, we have

$$
\begin{aligned}
S_\lambda &= S_{\lambda,0}(1+z)^{-5} \\
S_\nu &= S_{\nu,0}(1+z)^{-3}.
\end{aligned} \tag{10}
$$