Chris Malone
Cosmology: HW# 3
October 17, 2006

QUESTION 1

What is the light travel time of a quasar observed at redshift $z = 4.5$ in an Einstein-de Sitter Universe? Express your answer in terms of the Hubble Time.

In an Einstein-de Sitter universe, we know that the proper time at any redshift z is given by

$$t = \frac{2}{3}\frac{1}{H_0}\left(\frac{1}{1+z}\right)^{3/2} = \frac{2}{3}\left(\frac{1}{1+z}\right)^{3/2}\tau_0, \tag{1}$$

where τ_0 is the Hubble time. The total travel time is the difference between the time the photon was emitted, $t(z = 4.5)$, and the time it is observed, t_0, i.e. today. Therefore

$$\Delta t = \frac{2}{3}\left[1 - \left(\frac{1}{1+4.5}\right)^{3/2}\right]\tau_0 \approx 0.615\tau_0.$$

QUESTION 2

Show that in an Einstein-de Sitter Universe, angular diameter for fixed size reaches a minimum at redshift $z = 1.25$.

The angular diameter of an object is given as

$$\theta = \frac{l}{d_{ang}} \tag{2}$$

where

$$\begin{aligned} d_{ang} &= \frac{d_{lum}}{(1+z)^2} \\ d_{lum} &= \frac{2c}{H_0}\left((1+z) - \sqrt{1+z}\right). \end{aligned}$$

So we see from (2) that θ has a minimum when d_{ang} has a maximum. Basically, we need to look at

$$\begin{aligned} 0 &= \frac{d}{dz}\left(\frac{1}{1+z} - \frac{1}{(1+z)^{3/2}}\right) \\ &= \frac{1}{(1+z)^2} - \left(\frac{3}{2}\right)\frac{1}{(1+z)^{5/2}} \\ \sqrt{1+z} &= \frac{3}{2} \\ z &= \frac{9}{4} - 1 = \frac{5}{4} = 1.25. \end{aligned}$$

QUESTION 3

Consider a galaxy of bolometric luminosity $L = 10^{42}$ erg s^{-1} and diameter $d = 30$ kpc. Plot (1) the logarithm of the bolometric flux F and (2) the logarithm of the angular diameter θ of the galaxy versus $(1+z)$ over the redshift interval $z = 0$–20 for Friedmann cosmological models (i.e. $\Omega_\Lambda = 0$) with $\Omega_m = 0,\ 0.2,\ 1$, and 2.

First we have to derive[1] a formula for the angular diameter distance, d_{ang}, in terms of the comoving distance. We start with the FRW metric

$$ds^2 = dt^2 - \frac{R^2(t)}{c^2}\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] = 0, \tag{3}$$

where r is the comoving distance, R the scale factor, k the curvature, Ω the solid angle from observer to source, and the last equality comes from the fact that light travels along null geodesics. If we consider light travelling radially ($d\Omega = 0$) from a source at (t_e, r) to the observer at $(t_0, 0)$ we can rewrite this as

$$\int_{t_0}^{t_e} \frac{cdt}{R(t)} = -\int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = \int_r^0 \frac{dr'}{\sqrt{1-kr'^2}}, \tag{4}$$

where the "$-$" comes from the fact that as t increases, r decreases. We also know from the Friedmann equation that

$$\begin{aligned} \dot{R}^2 &= \frac{8\pi G\rho R^2}{3} - kc^2 = \frac{H_0^2 R_0^3 \Omega_0}{R} - H_0^2 R_0^2 (\Omega_0 - 1) \\ &= H_0^2 R_0^2 \left(\frac{\Omega_0 R_0}{R} - (\Omega_0 - 1)\right) \\ \Rightarrow \quad dt &= \frac{dR}{H_0 R_0 \sqrt{\frac{\Omega_0 R_0}{R} - (\Omega_0 - 1)}}, \end{aligned} \tag{5}$$

for a matter dominated universe. Plugging this back into (4) we have

$$\frac{c}{H_0 R_0} \int_{R(t_0)=R_0}^{R(t_e)} \frac{dR}{R\sqrt{\Omega_0 \frac{R_0}{R} - (\Omega_0 - 1)}} = \int_r^0 \frac{dr'}{\sqrt{1-kr'^2}}. \tag{6}$$

Now recalling that $a = R/R_0 = 1/(1+z)$, where z is the redshift of the source, the left hand side becomes

$$\frac{c}{H_0 R_0} \int_1^{1/(1+z)} \frac{da}{a\sqrt{\frac{\Omega_0}{a} - (\Omega_0 - 1)}}, \tag{7}$$

[1] I re-derive this here so I have a copy of the derivation. Skip to page 4 for the first graph.

Upon plugging back into (6) and rearranging integration order, we have

$$\int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = \frac{c}{H_0 R_0}\int_{1/(1+z)}^1 \frac{da}{a\sqrt{\frac{\Omega_0}{a}-(\Omega_0-1)}}. \tag{8}$$

Note that the left hand side takes on a different functional form for different values of k. For example, if k is negative we may factor out the constants and obtain the following:

$$\begin{aligned}
\int_0^r \frac{dr'}{\sqrt{1+\frac{H_0^2R_0^2}{c^2}(1-\Omega_0)r'^2}} &= \frac{c}{H_0R_0}\int_{1/(1+z)}^1 \frac{da}{a\sqrt{\frac{\Omega_0}{a}-(\Omega_0-1)}}\\
\frac{c}{H_0R_0\sqrt{1-\Omega_0}}\int_0^r \frac{dr'}{\sqrt{\frac{c^2}{H_0^2R_0^2(1-\Omega_0)}+r'^2}} &= \frac{c}{H_0R_0}\int_{1/(1+z)}^1 \frac{da}{a\sqrt{\frac{\Omega_0}{a}-(\Omega_0-1)}}\\
\sinh^{-1}\left(\frac{rH_0R_0\sqrt{1-\Omega_0}}{c}\right) &= \sqrt{1-\Omega_0}\int_{1/(1+z)}^1 \frac{da}{a\sqrt{\frac{\Omega_0}{a}-(\Omega_0-1)}}.
\end{aligned}$$

More generally, we may write the comoving distance as follows:

$$\begin{aligned}
k=-1\ (\Omega_0<1): &\qquad r = \frac{c}{H_0R_0\sqrt{1-\Omega_0}}\sinh\left[\chi(z,\Omega_0)\right]\\
k=0\ (\Omega_0=1): &\qquad r = \chi(z,1) = \frac{2c}{H_0}\left(1-\frac{1}{\sqrt{1+z}}\right)\\
k=+1\ (\Omega_0>1): &\qquad r = \frac{c}{H_0R_0\sqrt{\Omega_0-1}}\sin\left[\chi(z,\Omega_0)\right],
\end{aligned}$$

where we have plugged in the value of k as was done in (5). Now, the angular diameter distance is defined as

$$d_{ang} = \frac{l}{\theta}$$

where l is the physical size of an object which subtends an angle θ on the sky. In a comoving formalism we say that the physical size of the object is l/a and that the comoving distance to the object is r as derived above. We may therefore say that the angle subtended by the object is

$$\theta = \frac{l/a}{r}.$$

Comparing these two expressions, we see that

$$d_{ang} = ar = \frac{r}{1+z}. \tag{9}$$

Similarly, an expression for the luminosity distance can be derived:

$$d_{lum} \equiv \left(\frac{L}{4\pi F}\right)^{1/2} = r(1+z). \tag{10}$$

Finally, what we wish to plot is

$$\begin{aligned} F(z,\chi) &= \frac{L}{4\pi r^2(\chi)}\frac{1}{(1+z)^2} \\ \theta(z,\chi) &= \frac{d}{r(\chi)}(1+z) \end{aligned}$$

for various values of χ (essentially Ω_0) and z. First we need to perform the integral $\chi(z,\Omega_0)$ for the various cases. This is done analytically for all values of Ω_0 and the equations for r are:

$k=-1:$

$$r = \frac{c}{H_0 R_0\sqrt{1-\Omega_0}}\sinh\left(\ln\left[\frac{2\sqrt{1-\Omega_0}+2(1-\Omega_0)+\Omega_0}{2\sqrt{1-\Omega_0}\sqrt{\frac{1-\Omega_0}{(1+z)^2}+\frac{\Omega_0}{1+z}}+2\frac{1-\Omega_0}{1+z}+\Omega_0}\right]\right)$$

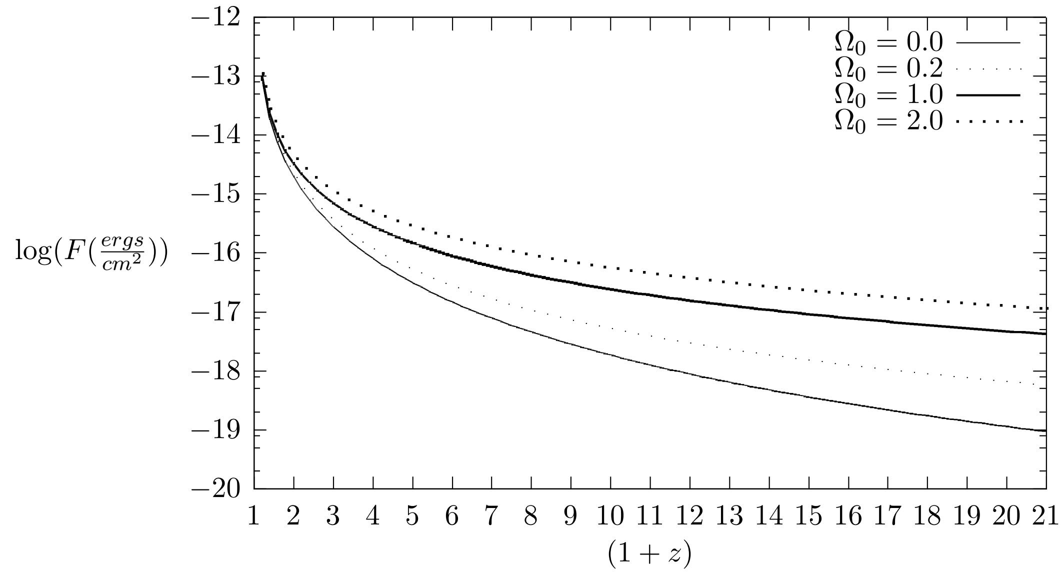
$k=0:$

$$r = \frac{2c}{H_0 R_0}\left(1-\frac{1}{\sqrt{1+z}}\right)$$

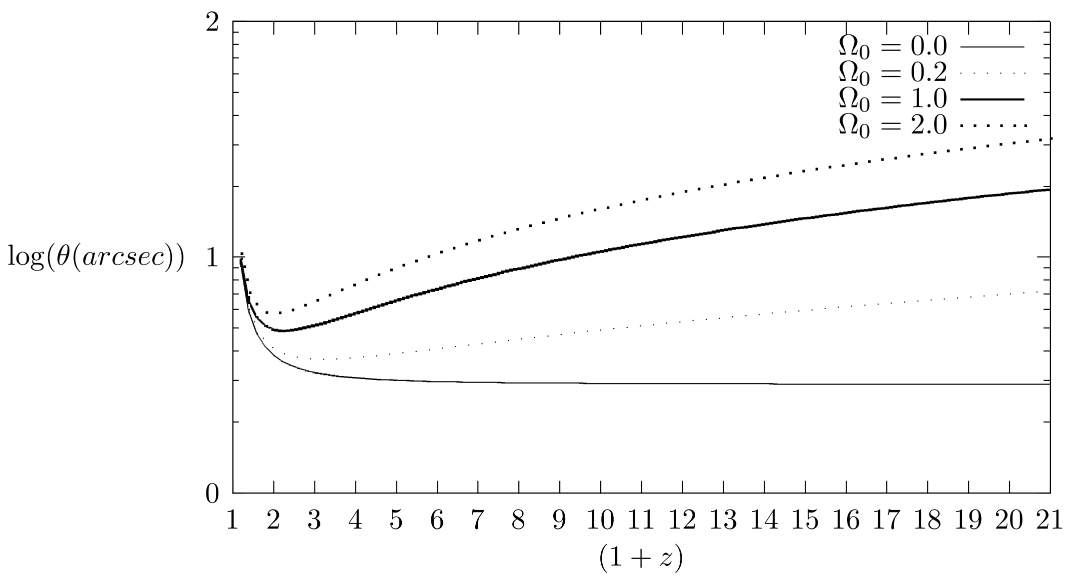
$k=+1:$

$$r = \frac{c}{H_0 R_0\sqrt{\Omega_0-1}}\sin\left[\sin^{-1}\left(1-2\frac{\Omega_0-1}{\Omega_0(1+z)}\right)-\sin^{-1}\left(\frac{2-\Omega_0}{\Omega_0}\right)\right].$$

These were now plugged into the previous definitions of the flux and angular diameter equations and plogged on a logscale. For these graphs, R_0 was set to unity.



$\log(F(\frac{ergs}{cm^2}))$
$(1+z)$
$\Omega_0 = 0.0$
$\Omega_0 = 0.2$
$\Omega_0 = 1.0$
$\Omega_0 = 2.0$



$\log(\theta(arcsec))$
$(1+z)$
$\Omega_0 = 0.0$
$\Omega_0 = 0.2$
$\Omega_0 = 1.0$
$\Omega_0 = 2.0$

QUESTION 4

Derive the integral relationship between radial coordinate and redshift for a Friedmann-Lemaitre model of matter density parameter Ω_m and vacuum energy density parameter Ω_Λ. Solve the relationship using Simpson's rule, and plot the luminosity distance and angular diameter distance versus redshift for redshifts $z = 0$–1000 for four choices of parameters: (1) $\Omega_m = 0$, $\Omega_\Lambda = 0$; (2) $\Omega_m = 1$, $\Omega_\Lambda = 0$; (3) $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$; (4) $\Omega_m = 0.5$, $\Omega_\Lambda = 0.5$.

To include a vacuum energy density we need to modify a few things from the previous problem. First, we take $\Omega_0 \to \Omega_m$. Next, we have to modify k and the dt from the Friedmann equation, (5):

$$k \quad \to \quad \frac{R_0^2 H_0^2}{c^2}(\Omega_m + \Omega_\Lambda - 1) \tag{11}$$

$$dt \quad \to \quad \frac{dR}{H_0^2 R_0^2 \sqrt{\Omega_m \left(\frac{R_0}{R}\right) + \Omega_\Lambda \left(\frac{R}{R_0}\right)^2 - (\Omega_m + \Omega_\Lambda - 1)}}. \tag{12}$$

If we perform the same manipulations as before, we have an integral relationship reminiscent of (8):

$$\int_0^r \frac{dr'}{\sqrt{1 - kr'^2}} = \frac{c}{H_0 R_0}\chi'(z, \Omega_m, \Omega_\Lambda) \equiv \frac{c}{H_0 R_0}\int_{1/(1+z)}^1 \frac{da}{a\sqrt{\frac{\Omega_m}{a} + \Omega_\Lambda a^2 - (\Omega_m + \Omega_\Lambda - 1)}}.$$

As before, the left hand side takes on different values for different values of k. We are only considering cases where either $k = 0$ or $k = -1$ and therefore:

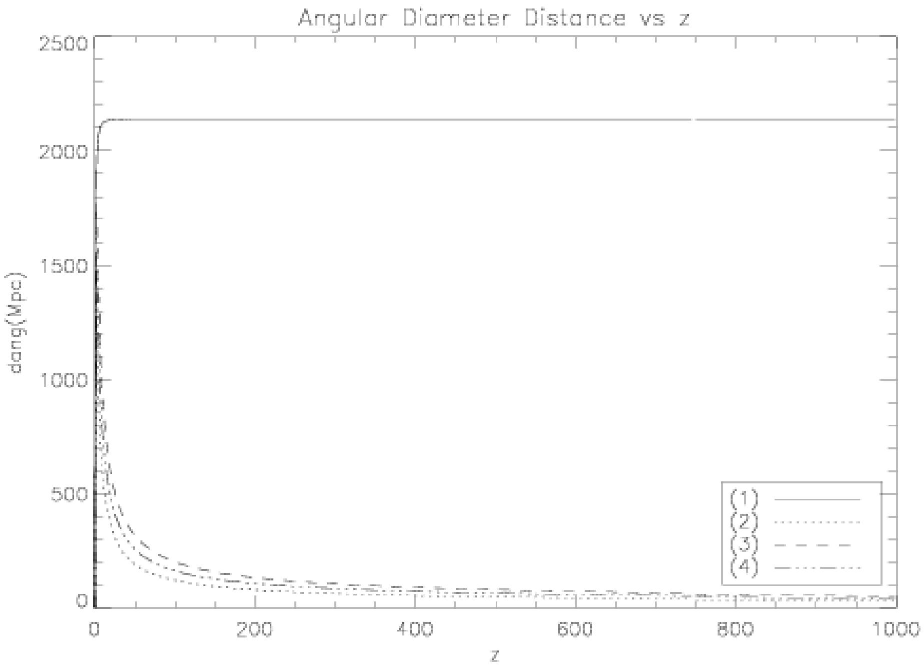
$$k = -1 \ (\Omega_m + \Omega_\Lambda < 1): \qquad r = \frac{c \sinh\left[\chi'(z, \Omega_m, \Omega_\Lambda)\sqrt{1 - \Omega_m - \Omega_\Lambda}\right]}{H_0 R_0 \sqrt{1 - \Omega_m - \Omega_\Lambda}}$$

$$k = 0 \ (\Omega_m + \Omega_\Lambda = 1): \qquad r = \frac{c}{H_0 R_0}\int_{1/(1+z)}^1 \frac{da}{\sqrt{\Omega_\Lambda a^4 + \Omega_m a}}.$$

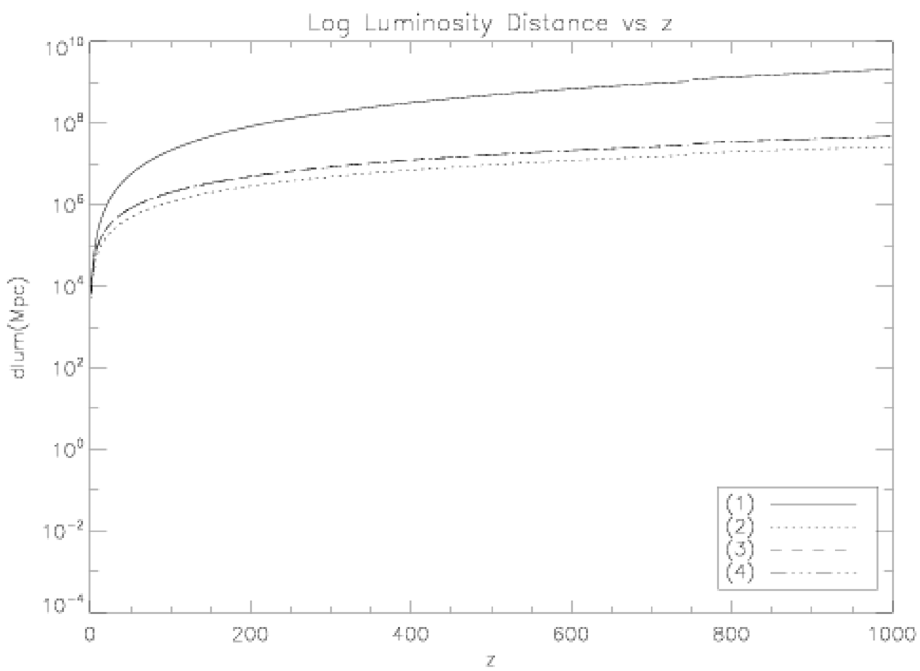
For the $k = -1$ case, our equation for r simplifies greatly because we have both Ω_m and Ω_Λ set to 0. We therefore have

$$r(\Omega_\Lambda = \Omega_m = 0) = \frac{c}{H_0 R_0}\sinh\left[\ln(1 + z)\right].$$

We are a plotting the angular diameter distance and the luminosity distance (given by (9) and (10)) versus redshift.



Angular Diameter Distance vs z
dang(Mpc)
z
(1)
(2)
(3)
(4)



Log Luminosity Distance vs z
dlum(Mpc)
z
(1)
(2)
(3)
(4)

QUESTION 5

Determine the proper and comoving volumes of the Universe bounded by $\Delta\Omega = 1$ arcmin^2 and redshift intervals of (1) $z = 1$–1.1, (2) $z = 3$–3.1, and (3) $z = 10$–10.1 for an Einstein-de Sitter cosmological model.

We know the relation between comoving volume, V_0, and the proper volume, V

$$V_0 = V(1+z)^3.$$

The next thing we need is to express the proper volume in terms of the solid angle, and redshift as was done in class for an Einstein-de Sitter universe:

$$V = \frac{4c^3\Omega\Delta z}{H_0^3}\left(\frac{1}{(1+z)^{9/2}} - \frac{2}{(1+z)^5} + \frac{1}{(1+z)^{11/2}}\right).$$

We use $H_0 = 70$ km s^{-1} Mpc^{-1} and $\Omega = 1$ arcmin$^2 \approx 8.46 \times 10^{-8}$ sr:

$$\begin{aligned}
z = 1 \text{ to } 1.1 & & \\
V &\approx 10.10 \text{ Mpc}^3\\
V_0 &\approx 80.80 \text{ Mpc}^3\\
z = 3 \text{ to } 3.1 & & \\
V &\approx 1.30 \text{ Mpc}^3\\
V_0 &\approx 83.24 \text{ Mpc}^3\\
z = 10 \text{ to } 10.1 & & \\
V &\approx 0.027 \text{ Mpc}^3\\
V_0 &\approx 35.62 \text{ Mpc}^3
\end{aligned}$$

APPENDIX:
PLOTTING WITH GNUPLOT

```
#
# ***** Program for plotting Flux and angular diameter
#    as function of redshift.
# *****

# change terminal to LaTeX files
set terminal latex

# set constants
c = 3e5  # km / s
h = 7e-2  # km / s / kpc
L = 1e43 # ergs / s
d = 3e1 # kpc
aspr = 3600.0*180.0/pi # '' / rad
cmpkpc = 3.08568025e21 # cm / kpc

# set plot properties for angular diameter
set logscale y
set xrange [1:21]
#set yrange [1e-5:1e-4]
set xtics 1
set format x "$%g$"
set format y "$%T$"
set xlabel "$(1+z)$"
set ylabel '$\log(\theta(arcsec))\qquad\qquad$'
set out "ang.tex"

# define functions

# ******************** R for Omega Less than One ***********************
ROLO(x,omega) = c*sinh(log((2*sqrt(1-omega) + 2*(1-omega) + omega)/(2*\
sqrt(1-omega)*sqrt((1-omega)/(x**2) + omega/x) + 2*(1-omega)/x + omega)))/(\
h*sqrt(1-omega))

# ******************** R for Omega Equal One **************************
ROEO(x) = 2*c*(1-1/sqrt(x))/h
```

```
# ******************* R for Omega Greater than One *******************
ROGO(x,omega) = c*sin(asin(1-(2*(omega-1))/(omega*x)) - asin((2-omega)/omega\
))/(h*sqrt(omega-1))


# ******************* Theta for Omega Less than One ******************
TOLO(x,omega) = d*x/ROLO(x,omega)*aspr

# ******************* Theta for Omega Equal One *********************
TOEO(x) = d*x/ROEO(x)*aspr

# ******************* Theta for Omega Greater than One ***************
TOGO(x,omega) = d*x/ROGO(x,omega)*aspr


# plot the angular diameter for the values required
plot TOLO(x,0.0) title '$\Omega_0=0.0$' lt 1, TOLO(x,0.2) title \
'$\Omega_0=0.2$' lt 2, TOEO(x) title '$\Omega_0=1.0$' lt 3, TOGO(x,2.0) \
title '$\Omega_0=2.0$' lt 4

# set plot properties for flux

set ylabel '$\log(F(\frac{ergs}{cm^2}))\qquad\qquad$'
set out "flux.tex"
#set yrange [1e23:1e31]

# ******************* Flux for Omega Less than One *******************
FOLO(x,omega) = (L/(4*pi))*(1/(x*ROLO(x,omega)*cmpkpc))**2

# ******************* Flux for Omega Equal One ***********************
FOEO(x) = (L/(4*pi))*(1/(x*ROEO(x)*cmpkpc))**2

# ******************* Flux for Omega Greater than One ****************
FOGO(x,omega) = (L/(4*pi))*(1/(x*ROGO(x,omega)*cmpkpc))**2


# plot the flux for the values required
plot FOLO(x,0.0) title '$\Omega_0=0.0$' lt 1, FOLO(x,0.2) title \
'$\Omega_0=0.2$' lt 2, FOEO(x) title '$\Omega_0=1.0$' lt 3, FOGO(x,2.0) title \
'$\Omega_0=2.0$' lt 4
```

```
exit
```

NUMERICAL INTEGRATION

```
FUNCTION calcr, z, om, ol

if N_params() eq 0 then begin
    print,'Syntax: result = ldist(z, Om = , Ol = ])'
    print,'Returns luminosity distance in Mpc'
    print,'Assumes R_0 = 0'
    return, 0.
endif

; Assuming H_0 = 70 km / s / Mpc
H0 = 70
c = 2.9979e5

zval = 1./(1.+z)

if (om + ol) EQ 1.0 then begin
; Simpson's Rule
;
; Equation we are integrating:          dx
;                               -------------------
;                               sqrt(Ol*x^4 + Om*x)
;
;     where Ol is the density fraction of the vacuum energy
;           Om is the density fraction of the matter
;

; Initial position
    x1 = 1./(1.+z)

    dstep = (1.0 - x1)/2.

    x2 = x1 + dstep
    x3 = 1.

; Integrand values at the various positions
```

```
    f1 = 1.0/sqrt(om*x1 + ol*x1^4)
    f2 = 1.0/sqrt(om*x2 + ol*x2^4)
    f3 = 1.0/sqrt(om*x3 + ol*x3^4)

    sum = (dstep/3.)*(f1 + 4.*f2 + f3)

    r = c*sum/H0
    return,r
endif

; The analytic expression
if (om + ol) LT 1 then begin
    r = c*sinh(alog(1+z))/H0
    return,r
endif

end


; make an array: z[i] = (0.0, 1.0, ... , 999.0, 1000.0)
z=findgen(1001)

; angular diameter distance
;
;              r
;   dang = ---------
;           (1 + z)
;

dang1 = calcr(z,0.0,0.0)/(1+z)
dang2 = calcr(z,1.0,0.0)/(1+z)
dang3 = calcr(z,0.3,0.7)/(1+z)
dang4 = calcr(z,0.5,0.5)/(1+z)

; luminosity distance
;
;
;   dlumm = r * (1 + z)
;
```

```
dlum1 = (1+z)*calcr(z,0.0,0.0)
dlum2 = (1+z)*calcr(z,1.0,0.0)
dlum3 = (1+z)*calcr(z,0.3,0.7)
dlum4 = (1+z)*calcr(z,0.3,0.7)

set_plot,'PS'
device, filename='dang.ps'

plot,z,dang1,xtitle='z',ytitle='dang(Mpc)'$
  ,title='Angular Diameter Distance vs z',linestyle=0
oplot,z,dang2,linestyle=1
oplot,z,dang3,linestyle=2
oplot,z,dang4,linestyle=4
legend,['(1)','(2)','(3)','(4)'],linestyle=[0,1,2,4],/right,/bottom

device, filename='dlum.ps'

plot,z,dlum1,/ylog,xtitle='z',ytitle='dlum(Mpc)'$
  ,title='Log Luminosity Distance vs z',linestyle=0
oplot,z,dlum2,linestyle=1
oplot,z,dlum3,linestyle=2
oplot,z,dlum4,linestyle=4
legend,['(1)','(2)','(3)','(4)'],linestyle=[0,1,2,4],/right,/bottom

end
```