Chris Malone
QFT: HW # 2
September 20, 2006

IA4.7
Consider the Poincaré group in 1 extra space dimension (D space, 1 time) for a massless particle. Interpret p^+ as the mass, and p^- as the energy.

a Show that the constraint $p^2 = 0$ gives the usual *nonrelativistic* expression for the energy.

b Show that the subgroup of the Poincaré group generated by all generators that commute with p^+ is the Galilean group (in D-1 space and time dimensions). Now nonrelativistic mass conservation is part of momentum conservation, and all the Galilean transformations are coordinate transformations. Also, positivity of the mass is related to positivity of the energy (see exercise **IA4.6**).

a The group ISO(4,1) in the lightcone basis, takes on the same form as ISO(3,1) except there is now a pentamomentum instead of 4-momentum where the new coordinate x^4 is another space coordinate. Therefore, the definition of p^+ and p^- are unaffected under this extension. Similarly,

$$p^2 = -2p^+p^- + (p^2)^2 + (p^3)^2 + (p^4)^2, \tag{1}$$

is an extension of the invariant p^2. Now, if we take p^+ to be the mass, p^- to be the energy, denote

$$\vec{p}^2 = (p^2)^2 + (p^3)^2 + (p^4)^2,$$

use the mass-shell condition $p^2 = 0$, and plug into (1) then we arrive at

$$p^2 = 0 = -2mE + \vec{p}^2. \tag{2}$$

Clearly, (2) is the expression for energy in the nonrelativisitic case.

b The generators of the Poincaré group are (P_a, J_{ab}), where in the lightcone basis a and b run over $+, -, 2, 3,$ and 4 (assuming, as was done in part **a**, we are in a 5D spacetime). We want to find, for *which* a and b are the following true:

$$[p^+, p_a] = [p^+, J_{ab}],$$

where

$$p^+ = -p_- = -\frac{1}{\sqrt{2}}\left(p^0 g_{00} - p^1 g_{11}\right) = \frac{1}{\sqrt{2}}\left(p_0 - p_1\right). \tag{3}$$

The momentum bracket is rather straightforward; all p operators are proportional to some derivative and we know that derivatives commute, therefore the p's should commute. Put another way, $[p_i, p_j] = 0$ for i and j running from 0 to 4, and $p^{\pm}$ are made up of $p^{0,1}$ and therefore should commute with all p's.

For the other bracket relation, we start from (3):

$$\begin{aligned}
[p^+, J_{ab}] &= \frac{1}{\sqrt{2}}[p_0 - p_1, x_a p_b - x_b p_a] \\
&= \frac{1}{\sqrt{2}}\Big([p_0, x_a]p_b + x_a[p_0, p_b] - [p_0, x_b]p_a - x_b[p_0, p_a] \\
&\qquad - [p_1, x_a]p_b - x_a[p_1, p_b] + [p_1, x_b]p_a + x_b[p_1, p_a]\Big) \\
&= \frac{1}{\sqrt{2}}\Big(\left([p_0, x_a] - [p_1, x_a]\right)p_b + \left([p_1, x_b] - [p_0, x_b]\right)p_a\Big). \qquad (4)
\end{aligned}$$

Now, we know that J_{ab} is anti-symmetric which means that when $a = b$, (4) obviously commutes. The only other case to study is when *both* of the coefficients of p_a and p_b in (4) disappear, namely

$$[p_0, x_a] - [p_1, x_a] = 0$$

AND

$$[p_1, x_b] - [p_0, x_b] = 0.$$

The first expression tells us that a cannot be 0 or 1; this, in our lightcone basis, tells us that a cannot be "+" or "-." Similarly, the second expression states that b cannot be "+" or "-." We are therefore restricted to the 3×3 antisymmetric submatrix, $\widetilde{J}_{ab}$, of J_{ab} whose components are entirely spatial:

$$\widetilde{J}_{ab} = \begin{pmatrix} 0 & J_{23} & J_{24} \\ -J_{23} & 0 & J_{34} \\ -J_{24} & -J_{34} & 0 \end{pmatrix}. \tag{5}$$

We now have the translation (momentum) and the rotation (angular momentum) generators of a (3-space, 1-time) Galilean group.

IA6.5

The conformal group for Euclidean space (or any spacetime signature) can be obtained by the same construction. Consider the special case of D=2 for these SO(D+1,1) transformations. (This is a subgroup of the 2D superconformal group: See exercise **IA6.1**.) Use complex coordinates for the two "physical" dimensions:

$$z = \frac{1}{\sqrt{2}}(x^1 + ix^2)$$

a Show that the inversion is

$$z \leftrightarrow -\frac{1}{z^*}$$

b Show that the conformal boost is (using a complex number also for the boost vector)

$$z \rightarrow \frac{z}{1 + v^* z}$$

a We begin by assuming that the given transformation is indeed an inversion, and rewrite it to get into the usual format of an inversion.

$$\begin{aligned} z \leftrightarrow -\frac{1}{z^*} &= -\frac{z}{z^* z} \\ &= -\frac{\frac{1}{\sqrt{2}}(x^1 + ix^2)}{\frac{1}{2}\Big((x^1)^2 + (x^2)^2\Big)}, \end{aligned}$$

which is of the form

$$\begin{pmatrix} \frac{x^1}{\sqrt{2}} \\ \\ \frac{x^2}{\sqrt{2}} \end{pmatrix} \longleftrightarrow -\frac{2}{\mathbf{x}^2} \begin{pmatrix} \frac{x^1}{\sqrt{2}} \\ \\ \frac{x^2}{\sqrt{2}} \end{pmatrix},$$

in the $\begin{pmatrix} 1 \\ i \end{pmatrix}$ basis.

b Following a similar procedure as part **a**

$$\begin{aligned} z \rightarrow \frac{z}{1 + v^* z} &= \frac{z(1 + vz^*)}{(1 + v^* z)(1 + vz^*)} \\ &= \frac{z + v(zz^*)}{1 + v^* z + vz^* + (v^* v)(z^* z)}, \end{aligned}$$

which is of the form

$$\begin{pmatrix} \frac{x^1}{\sqrt{2}} \\ \\ \frac{x^2}{\sqrt{2}} \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{\frac{x^1}{\sqrt{2}}+\frac{1}{2}\mathbf{x}v^1}{1+\mathbf{v}\cdot\mathbf{x}+\frac{1}{2}\mathbf{x}^2\mathbf{v}^2} \\ \\ \frac{\frac{x^2}{\sqrt{2}}-\frac{i}{2}\mathbf{x}^2v^2}{1+\mathbf{v}\cdot\mathbf{x}+\frac{1}{2}\mathbf{x}^2\mathbf{v}^2} \end{pmatrix}$$

in the same basis as before.