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QFT: HW # 8
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IIIA3.1
Consider the action

$$S[\phi] = \int dt d^{D-1}x \, \left[-\frac{1}{2}\dot{\phi}^2 + V(\phi) \right]$$

for potential $V(\phi)$ (a function, not a functional).

a Find the field equations.

b Assume $V(\phi) = \lambda\phi^n$ for some positive integer n and constant, *dimensionless* λ, in units $\hbar = c = 1$. Use dimensional analysis to relate n and D (of course, also a positive integer), and list all paired possibilities of (n, D).

a To find the field equations, we simply need to set the variation of the action to zero:

$$\begin{aligned} \delta S[\phi] = 0 &= \delta \left(\int dt d^{D-1}x \, \left[-\frac{1}{2}\dot{\phi}^2 + V(\phi) \right] \right) \\ &= \int dt d^{D-1}x \, \left[-\dot{\phi}\delta\dot{\phi} + \delta V(\phi) \right] \\ &= \int dt d^{D-1}x \, \left[\ddot{\phi}\delta\phi + \frac{\partial V(\phi)}{\partial \phi}\delta\phi \right] \\ &= \int dt d^{D-1}x \, \delta\phi \left[\ddot{\phi} + \frac{\partial V(\phi)}{\partial \phi} \right] \\ \therefore & \quad \frac{\delta S[\phi]}{\delta \phi} = \ddot{\phi} + \frac{\partial V(\phi)}{\partial \phi} = 0 \\ \Rightarrow & \quad \ddot{\phi} = -\frac{\partial V(\phi)}{\partial \phi} \end{aligned}$$

b In the units where $\hbar$ and c are dimensionless, our action is dimensionless. Therefore the integral

$$\int dt d^{D-1}x \, \phi^n$$

is also dimensionless.

IIIA4.1

Let's consider the semiclassical interpretation of a charged particle as described by a complex scalar field ψ, with Lagrangian

$$L = \frac{1}{2}\left(|\nabla\psi|^2 + m^2|\psi|^2\right) \tag{1}$$

a Use the semiclassical expansion in $\hbar$ defined by

$$\nabla \to \hbar\partial + iqA, \qquad \psi \to \sqrt{\rho}e^{-iS/\hbar}$$

Find the Lagrangian in terms of ρ and S (and the background field A), order-by-order in $\hbar$ (in this case, just $\hbar^0$ and $\hbar^2$).

b Take the semiclassical limit by dropping the $\hbar^2$ term in L, to find

$$L \to \rho\frac{1}{2}\left[(-\partial S + qA)^2 + m^2\right]$$

Vary with respect to S and ρ to find the equations of motion. Defining

$$p \equiv -\partial S$$

show that these field equations can be interpreted as the mass-shell condition and current conservation. Show that A couples to this current by varying L with respect to A.

a Using this replacement, we can rewrite (1) as:

$$\begin{aligned}
L &= \frac{1}{2}\left(\left|(\hbar\partial + iqA)(\sqrt{\rho}e^{-iS/\hbar})\right|^2 + m^2\left|\sqrt{\rho}e^{-iS/\hbar}\right|^2\right) \\
&= \frac{1}{2}\left(\left|(\hbar\partial + iqA)(\sqrt{\rho}e^{-iS/\hbar})\right|^2 + m^2\rho\right) \\
&= \frac{1}{2}\left(\left|\left[\frac{\hbar\partial\rho}{2\sqrt{\rho}} + i\sqrt{\rho}(qA - \partial S)\right]e^{-iS/\hbar}\right|^2 + m^2\rho\right) \\
&= \frac{1}{2}\left(\frac{\hbar^2}{4\rho}(\partial\rho)^2 + \rho(qA - \partial S)^2 + m^2\rho\right) \\
L &= \frac{\rho}{2}\left(\frac{1}{4}\left(\frac{\partial\rho}{\rho}\right)^2\hbar^2 + \left[(qA - \partial S)^2 + m^2\right]\hbar^0\right).
\end{aligned} \tag{2}$$

b Now, if we drop the terms in (2) of order $\hbar^2$ clearly we have

$$L = \frac{\rho}{2}\left[(-\partial S + qA)^2 + m^2\right]. \tag{3}$$

To get the equations of motion, we need to solve the Euler-Lagrange equations:

$$\frac{\partial L}{\partial \rho} - \partial \frac{\partial L}{\partial(\partial \rho)} = 0, \quad \frac{\partial L}{\partial S} - \partial \frac{\partial L}{\partial(\partial S)} = 0.$$

We see that (3) has no $\partial\rho$ term and also no S term. Therefore we can neglect the partial derivatives with respect to these variables. The other terms become:

$$\begin{aligned}
\frac{\partial L}{\partial \rho} &= \frac{1}{2}\left[(-\partial S + qA)^2 + m^2\right] = 0 \\
&\Rightarrow (-\partial S + qA)^2 + m^2 = 0 \qquad (4)\\
-\partial \frac{\partial L}{\partial(\partial S)} &= \partial\left(\rho\left[-\partial S + qA\right]\right) = 0 \\
&\Rightarrow \partial\left(-\rho\partial S + q\rho A\right) = 0. \qquad (5)
\end{aligned}$$

Now, if we make the substitution for $-\partial S$ as stated in the problem, we see (4) becomes

$$(p + qA)^2 + m^2 = 0 \tag{6}$$

which is exactly the mass shell condition, where the total momentum includes terms from the gauge field. Similarly, plugging into (5) gives

$$\partial\left(\rho p + q\rho A\right) = \partial J = 0 \tag{7}$$

or that the current (again including terms due to the gauge field) is a conserved quantity. For the A coupling we have

$$\frac{\partial L}{\partial A} = q\rho\left(p + qA\right) = qJ = 0. \tag{8}$$

IIIA4.3

By plugging in the appropriate expressions in terms of A_a (and repeatedly integrating by parts), show that all of the above expressions for the electromagnetism action can be written as

$$S_A = -\int dx \frac{1}{4e^2}\left[A \cdot \Box A + (\partial \cdot A)^2\right]$$

If we start from the expression in the book:

$$S_A = \int dx\ \frac{1}{8e^2} F^{ab} F_{ab} + A^a J_a$$

where $J^b = \frac{1}{2e^2}\partial_a F^{ab}$ is the current density and expand the field tensor we have

$$\begin{aligned}
S_A &= \int dx\ \frac{1}{8e^2}\left(\partial^a A^b - \partial^b A^a\right)\left(\partial_a A_b - \partial_b A_a\right) + \frac{1}{2e^2}\int dx\ A^a \partial^\alpha \left(\partial_\alpha A_a - \partial_a A_\alpha\right) \\
&= \int dx\ \frac{1}{8e^2}\left(\partial^a A^b \partial_a A_b - \partial^a A^b \partial_b A_a - \partial^b A^a \partial_a A_b + \partial^b A^a \partial_b A_b\right) \\
&\quad + \frac{1}{2e^2}\int dx\ A^a \partial^\alpha \left(\partial_\alpha A_a - \partial_a A_\alpha\right) \\
&= \frac{1}{4e^2}\int dx\ \left[(\partial^a A^b)(\partial_a A_b) - (\partial^a A^b)(\partial_b A_a) + 2\left\{A^a \partial^\alpha \left(\partial_\alpha A_a - \partial_a A_\alpha\right)\right\}\right] \\
&= \frac{1}{4e^2}\int dx\ \left[-A\cdot\Box A - (\partial^a A^b)(\partial_b A_a) + 2A\cdot\Box A - 2A^a\partial^\alpha\partial_a A_\alpha\right] \\
&= \frac{1}{4e^2}\int dx\ \left[A\cdot\Box A - (\partial^a A^b)(\partial_b A_a) - 2A^a\partial^\alpha\partial_a A_\alpha\right]
\end{aligned}$$