Equation of State

Chapter 3 Prialnik

Chapter 5 Krumholtz and Glatzmaier

Chapter 3 Pols

Usually, though not universally[*], the independent variables in a stellar evolution model are radius (or mass), density, and temperature. In addition, the matter is characterized, at any point in time, by a composition $\{X_i\}$.

The "equation of state" tells how such thermodynamic quantities as the pressure and internal energy are given as functions of these variables.

In general, this is a solved problem in physics, although often a tedious one, that can be handled by the computer.

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[*] especially in multi-dimensional codes

Stars are spheres of mostly ionized gas

The criterion for being a gas (as opposed to a liquid or solid) is that the thermal energy of a pair of ions substantially exceed their electrical potential energy, i.e.

$$\mathrm{kT} >> \frac{Z_1 Z_2 e^2}{r_{12}}$$

where the typical separation depends on the density and ion mass

$$\rho \approx \frac{3m_{ion}}{4\pi r_{12}^3} \Rightarrow r_{12} = \left(\frac{3m_{ion}}{4\pi\rho}\right)^{1/3} \qquad m_{ion} = Am_H = \frac{A}{N_A}$$

where A is the atomic mass of the nucleus - e.g., 12 for carbon, and $\mathrm{N}_A = 6.02 \times 10^{23}$ *particles / Mole* is the inverse weight of an Hatom

$$\mathrm{kT} >> \frac{Z_1 Z_2 e^2}{r_{12}} \approx Z_1 Z_2 e^2 \left(\frac{4\pi\rho N_A}{3A}\right)^{1/3}$$

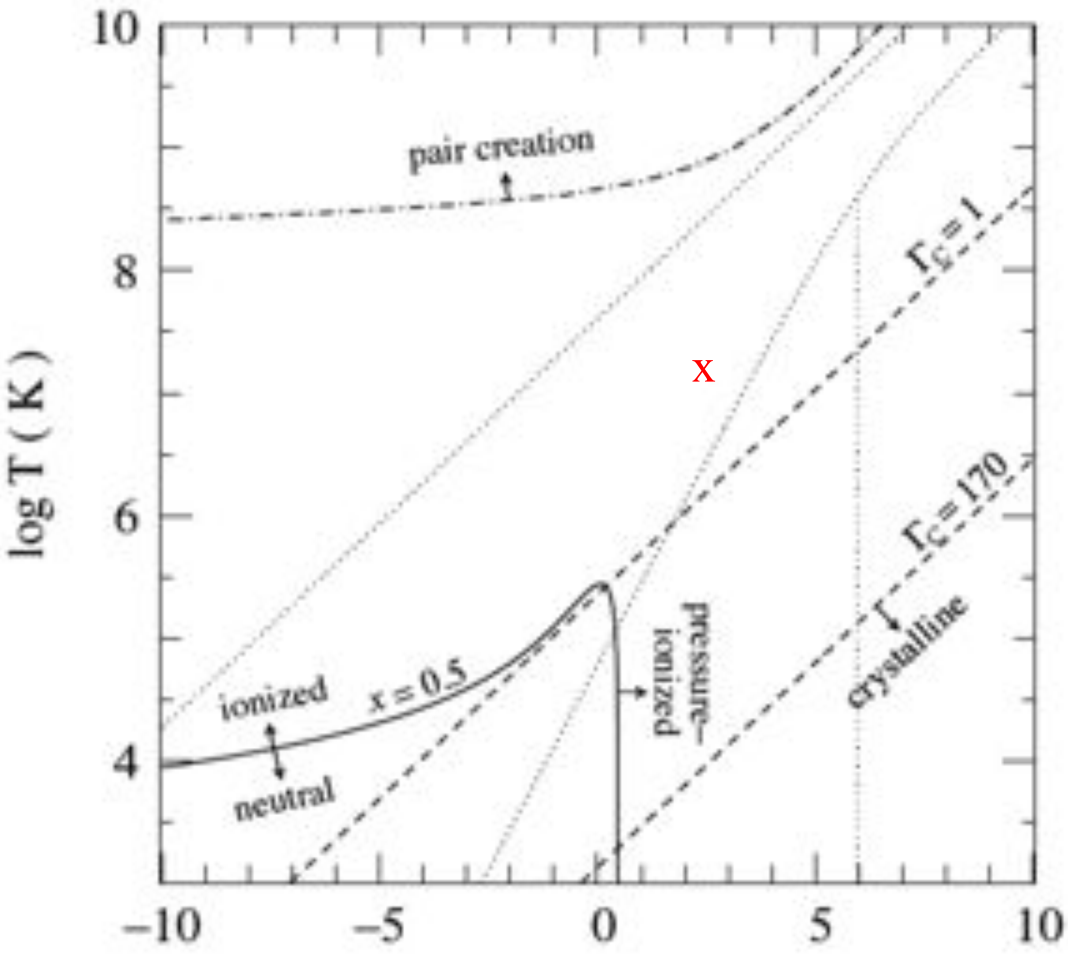
$$T >> \left(\frac{Z_1 Z_2 e^2}{k}\right)\left(\frac{4\pi N_A}{3A}\right)^{1/3} \rho^{1/3}$$

e.g., for a single component gas with Z, A

$$T >> \left(\frac{Z^2 e^2}{k}\right)\left(\frac{4\pi N_A}{3A}\right)^{1/3} \rho^{1/3} = 2.3\times 10^5 \left(\frac{Z^2}{A^{1/3}}\right)\rho^{1/3} \text{ Kelvin}$$

Generally stars satisfy this condition. The mean density of the sun is 1.4, the mean temperature several million (e.g. from the Virial theorem). The center of the sun (150 g cm^{-3};15 MK) satisfies this condition. The solar surface layers where T is as small as 5000 K have a very low density and also satisfy this condition. Heavier main sequence stars are hotter and less dense. Even in very low mass main sequnece stars, this condition is satisfied, though not by a large margin. It is important that nuclei with high Z - like iron (Z= 26) - are rare in stars.

Pols (p 39)



log ρ

Figure 3.7. The equation of state in the ρ, T plane for a pure hydrogen gas. The dotted lines are the borders, also shown in Fig. 3.4, between regions where radiation, ideal gas and degenerate electrons dominate the pressure. The solid line shows where the ionization fraction of hydrogen is 0.5 according to the Saha equation, and where hydrogen becomes pressure-ionized at high density. The dashed lines show where the Coulomb interaction parameter Γ_C equals 1, above which Coulomb interactions become important, and where $\Gamma_C = 170$, above which the ions form a crystralline lattice. Above the dash-dotted line e^+e^- pairs play an important role in stellar interiors.

Is the gas ionized?

If one applies the Saha equation to the center of the sun (ρ=160 $\text{g}\,\text{cm}^{-3}$; T = 15.7×10^{6} K), one finds, perhaps surprisingly, that not all the hydrogen is ionized.

$$\frac{y^2}{(1-y)} = \frac{1}{\rho N_A X_H}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-13.6/kT}$$

$$= \frac{1}{(160)\left(6.02\times10^{23}\right)(0.35)}\left(2.41\times10^{15}\right)\left(1.57\times10^{7}\right)^{3/2} e^{-13.6/1353.}$$

$$= 4.4$$

where 1353 eV is 15.7 MK turned into eV (the temperature at the center of the sun is about a kilovolt - 1.35 keV). Note that, very unlike the photosphere, the exponential doesn't do anything here. It's just $\approx 1.$ Solving for y we get

$$y = \frac{n(H\,II)}{n(H\,I)} = 0.84$$

16% of the hydrogen is neutral!

Pressure ionization

But the hydrogen is all ionized due to an effect called "pressure ionization". As the average inter-ion spacing becomes comparable to atomic dimensions, the energy required to ionize an atom is reduced.

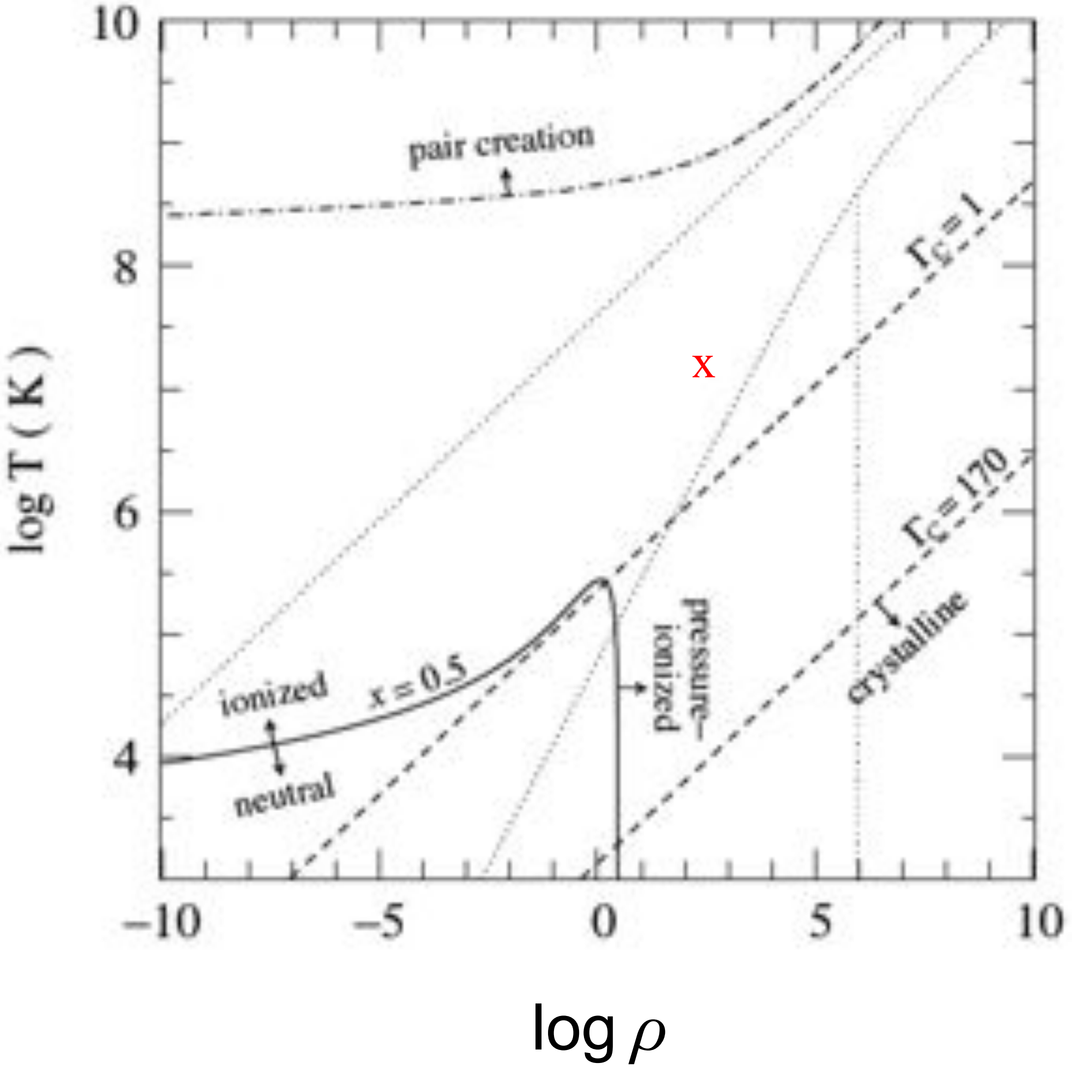
The average ion spacing for hydrogen at 160 g cm^{-3} is

$$d \approx \left(\frac{3}{4\pi n_H}\right)^{1/3} = \left(\frac{3}{4\pi\rho N_A X_H}\right)^{1/3} = 1.9\times10^{-9}\,\text{cm if } X_H = 0.35$$

The radius of the ground state of the H atom is 5 $\times 10^{-9}$ cm, so H is ionized.

Actually since the temperature is not zero the hydrogen is ionized due to a combination of thermal and density effects at an even lower density. The effect of the background density is to decrease the 13.6 eV ionization potential of H. This shows a limitation of the simplest Saha equation.

Pols (p 39)



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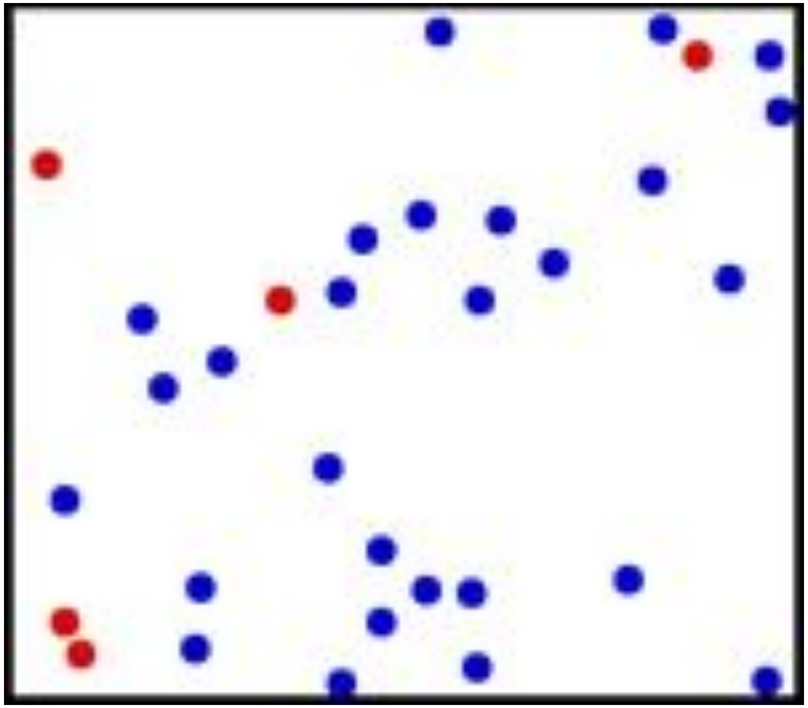
Non perfect gas effects:

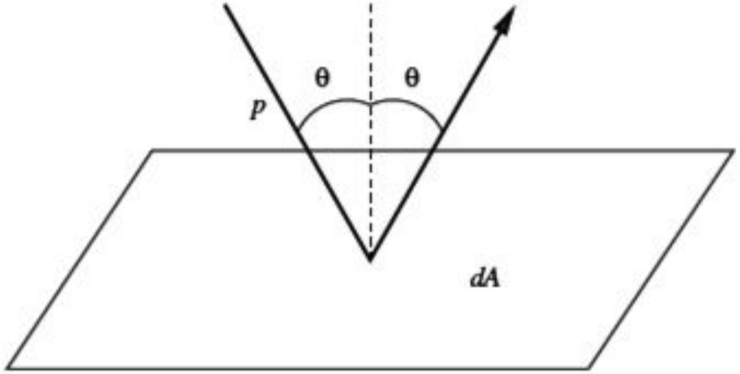
- Even in low mass main sequence stars there may be appreciable corrections, these so called non-ideal or “Coulomb corrections” need to be accounted for in accurate models.
- The Coulomb corrections can be appreciable in white dwarfs and old cold white dwarfs may crystalize
- The Chandrasekhar mass is substantially altered (reduced) by Coulomb corrections

  .

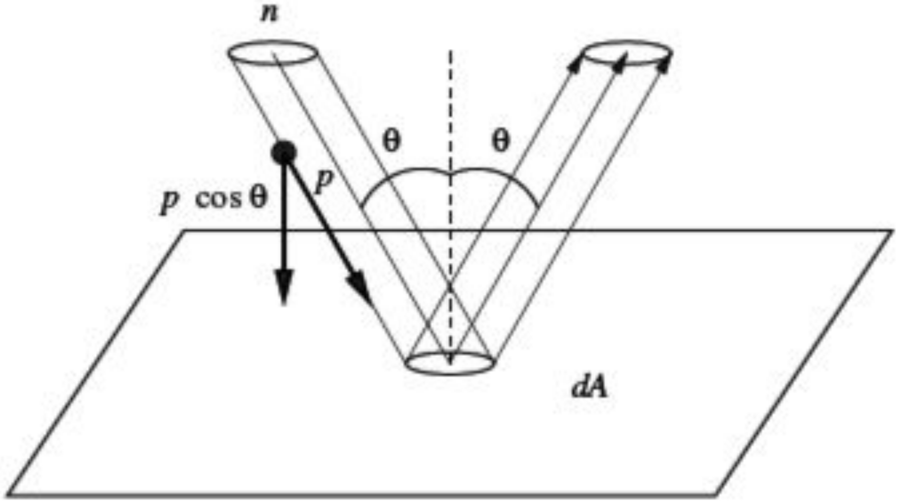
The pressure integral



http://intro.chem.okstate.edu/1314f00/laboratory/glp.htm



Consider a single particle impacting at angle θ. The momentum transferred to the plane is 2 p Cos θ

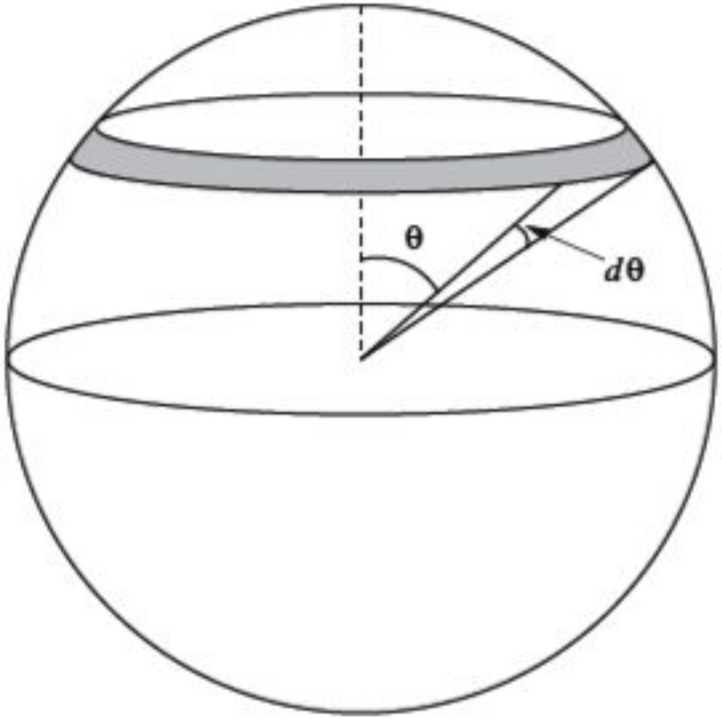


Now consider a beam of particles with the same momentum and angle with number density n.

The incident flux is
nv Cos θ per unit area
and the momentum transfer now is

$$\frac{d^2 p_{surf}}{dt\,dA} = 2 n v\, p\, Cos^2(\theta)$$

Only a fraction of the particles impact dA per unit time. If θ is 90° none do. If θ is 0° they all do



Continuing to assume all particles have the same momentum but an isotropic distribution, the the fraction of particles at angle θ is just the fraction of the solid angle that lies between θ and θ+dθ. The area of the strip is 2 π sin θ dθ and the solid angle of the entire sphere is 4 π, so we have

$$\frac{dn(\theta)}{d\theta} = \frac{1}{2} n\, Sin(\theta)$$

Combining terms:

$$\frac{d^2 p_{surf}}{dt\,dA} = npv \int_0^{\pi/2} Cos^2\theta\ Sin\theta\ d\theta$$

$$= -npv \int_1^0 Cos^2\theta\ d(Cos\theta) = npv \int_0^1 Cos^2\theta\ d(Cos\theta)$$

$$= npv \left.\frac{x^3}{3}\right)_0^1 = \frac{1}{3}\,n\,p\,v$$

if p is a constant but the angular distribution is isotropic

For a distribution of momenta

$$\boxed{P \equiv \frac{d^2 p_{surf}}{dt\,dA} = \frac{1}{3}\int_0^\infty \frac{dn(p)}{dp}\,p\ v\ dp}$$

This is called the “pressure integral”. It is the momentum transferred to the surface per unit time per unit area

So far we have said nothing about the particles

To evaluate the pressure integral one needs a description of dn(p)/dp. Because different particles have different distributions of momentum for a given temperature and density there are different kinds of pressure. “Distribution functions” are a way of counting energy states in the gas and the many ways the particles can be arrayed (distributed) in those states.

Particles can be distinguishable (ideal gas) or indistinguishable (radiation and QM particles). They can be packed many per state, or only one (Pauli exclusion principle for Fermions like neutrons, protons and electrons).

E.g.

Consider the ways 2 particles A and B can be put into just 2 states.

http://eiffel.ps.uci.edu/cyu/p214A/LectureNotes/Lecture13/lecture13.pdf

State 1	State2		
AB		4 states MB statistics (distinguishable)	e.g., by x, y, x p_x, p_y, p_z
	AB		
A	B		
B	A		
or			
AB		3 states BE statistics (indistinguishable)	
	AB		
A or B	A or B		
or			
A or B	B or A	1 state FD statistics (one indistinguishable particle per state)	

Statistical mechanics is a way of describing a system of many particles given certain macroscopic quantities such as the temperature and total number of particles.

The basic entities are “microstates” which might be defined for a classic system as the (x,y,z) and (p_x, p_y, p_z) coordinates of every particle or in quantum mechanics by the allowed states subject to given quantum numbers and restrictions.

A “macrostate” is a state of of the system where the distribution of particles over the energy levels is specified subject to conservation of energy. It contains many microstates.

The disorder number Ω is the number of microstates available to a given macrostate. It can be a big number, but varies with how the available energy is distributed. A fundamental assumption of statistical mechanics is that the system will distribute itself “probabilistically”, spending most of the time in a macrostate that has high probability (e.g., 100 coins flipped are more likely to distribute around 50 heads and tails than 100 heads). This most probable state is “equilibrium”. It is a state of maximum “entropy”.

Operationally then statistical mechanics is a way of counting microstates, subject to different restrictions, and enumerating their probability.

Classical particles are “distinguishable” by their coordinates which, being continuous, form an infinite set. However they can be “coarse binned” into groups between r and r + dr, p and p+ dp.

Quantum particles, when examined at the level of their wavelength are indistinguishable. The wave functions cannot be distinguished

Spin ½ particles (fermions) are limited to 2 per quantum mechanical shell. This has to do with the symmetry properties of the wave function. Exchanging two particles changes the sign of the wave function

Spin 1 particles (bosons) are not limited. Any number can go into a state. For relativistic bosons, e.g., photons, particle number is not conserved.

http://www.spms.ntu.edu.sg/PAP/courseware/statmech.pdf

DISTRIBUTION FUNCTIONS

The “distribution function” gives the likely number of particles in a given energy state, ε_s, in the equilibrium macrostate

$$n_s = N \frac{e^{-\varepsilon_s/kT}}{\sum_r e^{-\varepsilon_r/kT}}$$

Maxwell Bolzmann

$$= \frac{1}{e^{\varepsilon_s/kT} - 1}$$

Bose-Einstein
($\mu = 0$ for photons
because N not fixed)

$$= \frac{1}{e^{(\varepsilon_s - \mu)/kT} + 1}$$

Fermi-Dirac

When combined with a volume factor for phase space $4\pi p^2$ dp and an inherent degeneracy factor (2 for spin states for Fermions; 2 for polarization for photons). One has.

$$\frac{dn}{dp} \propto 4\pi p^2 e^{-\varepsilon/kT}$$ MB

$$= \frac{1}{h^3}\frac{(2)(4\pi p^2)}{e^{\varepsilon/kT}-1}$$ BE

$$= \frac{1}{h^3}\frac{(2)(4\pi p^2)}{e^{(\varepsilon-\mu)/kT}+1}$$ FD

The factor $\frac{1}{h^3}$ in the FD and BE distributions reflects the fact that they describe quantum mechanical systems which obey

$$V\,\Delta^3 p \ge h^3$$

where V is the volume per particle (actually per pair of particles for Fermions). The volume per particle is also 1/n, where n is the number density of particles (if particles are conserved).

$$dN \propto \frac{4\pi p^2 dp}{\Delta^3 p} = \frac{4\pi p^2 dp}{h^3} V$$

$$dn = \frac{dN}{V} = \frac{4\pi p^2 dp}{h^3}$$

Ideal gas – Maxwell-Boltzmann statistics $\varepsilon = p^2/2m$

$$\frac{dn(p)}{dp} \propto 4\pi p^2 \exp(-p^2/2mkT)$$

Normalization comes from requiring that the total density of particles per cm^3 be n, i.e., for C a constant

$$n = C\,4\pi \int_0^\infty p^2 \exp(-p^2/2mkT)\,dp$$

$$= C\,4\pi\,(2mkT)^{3/2} \int_0^\infty q^2 e^{-q^2}\,dq \qquad q \equiv \frac{p}{\sqrt{2mkT}}$$

$$= C\,(2\pi mkT)^{3/2}$$ (the integral is equal to $\frac{\sqrt{\pi}}{4}$)

So C = $\frac{n}{(2\pi mkT)^{3/2}}$

This is the Maxwell Boltzmann distribution

and

$$\frac{dn(p)}{dp} = \frac{4\pi n}{(2\pi mkT)^{3/2}} p^2 \exp(-p^2/(2mkT))$$

Putting it all together:

$$P = \frac{1}{3}\int_0^\infty \left[\frac{4n}{\pi^{1/2}\left(2mkT\right)^{3/2}} p^2 \; e^{-p^2/(2mkT)} \right] p \, \frac{p}{m} \, dp$$

$$= \frac{4n}{3\pi^{1/2}m} (2mkT) \int_0^\infty q^4 \, e^{-q^2} \, dq \qquad\qquad q = \frac{p}{\sqrt{2mkT}}$$

$$= \frac{4n}{3\pi^{1/2}m} (2mkT) \left(\frac{3\sqrt{\pi}}{8} \right) \quad = nkT = P_{ideal}$$

Ingredients:

1) Pressure integral
2) Distribution function – $e^{-E/kT}$
3) Phase space - $4\pi p^2 dp$
4) Normalization - number of particles conserved

What is n in $P = nkT$?

The hard part in practice to evaluating ideal gas pressure actually comes from counting the number of particles, n.

Electrons and ions must be counted separately. If the gas is fully ionized, as is generally the case deep in the stellar interior (kT >> ionization energy), then charge neutrality requires that the number of electrons (freely moving) equals the number of protons (bound in nuclei).

It is more problematic what to do when the gas is partly ionized as it frequently is near the stellar surface. Then a Saha equation (or set of Saha equations) must be solved.

Example: *What is the pressure of a fully ionized gas composed of 75% H and 25% He at a temperature of 10^7 K and a density of 100 g cm^{-3} (similar to solar center when the sun was born)?*

In each cm^3 of gas there are 75 gm of hydrogen and 25 gram of helium.

In 75 gm of hydrogen there are

$$\frac{75 \text{ gm}}{m_{proton}} = 75 \text{ N}_A = 75 \times 6.02 \times 10^{23} = 4.52 \times 10^{25} \text{ protons}$$

In 25 gram of helium there are (the helium nucleus weighs approximately 4 m_{proton})

$$\frac{25 \text{ gm}}{4 m_{proton}} = 6.25 \text{ N}_A = 6.25 \times 6.02 \times 10^{23} = 3.76 \times 10^{24} \text{ helium nuclei}$$

In addition, each hydrogen contributes to the ionized plasma 1 electron and each helium contributes two

$$n = 4.52 \times 10^{25} + \overbrace{4.52 \times 10^{25}}^{\text{electrons}} + 3.76 \times 10^{24} + \overbrace{2(3.76 \times 10^{24})}^{\text{electrons}}$$

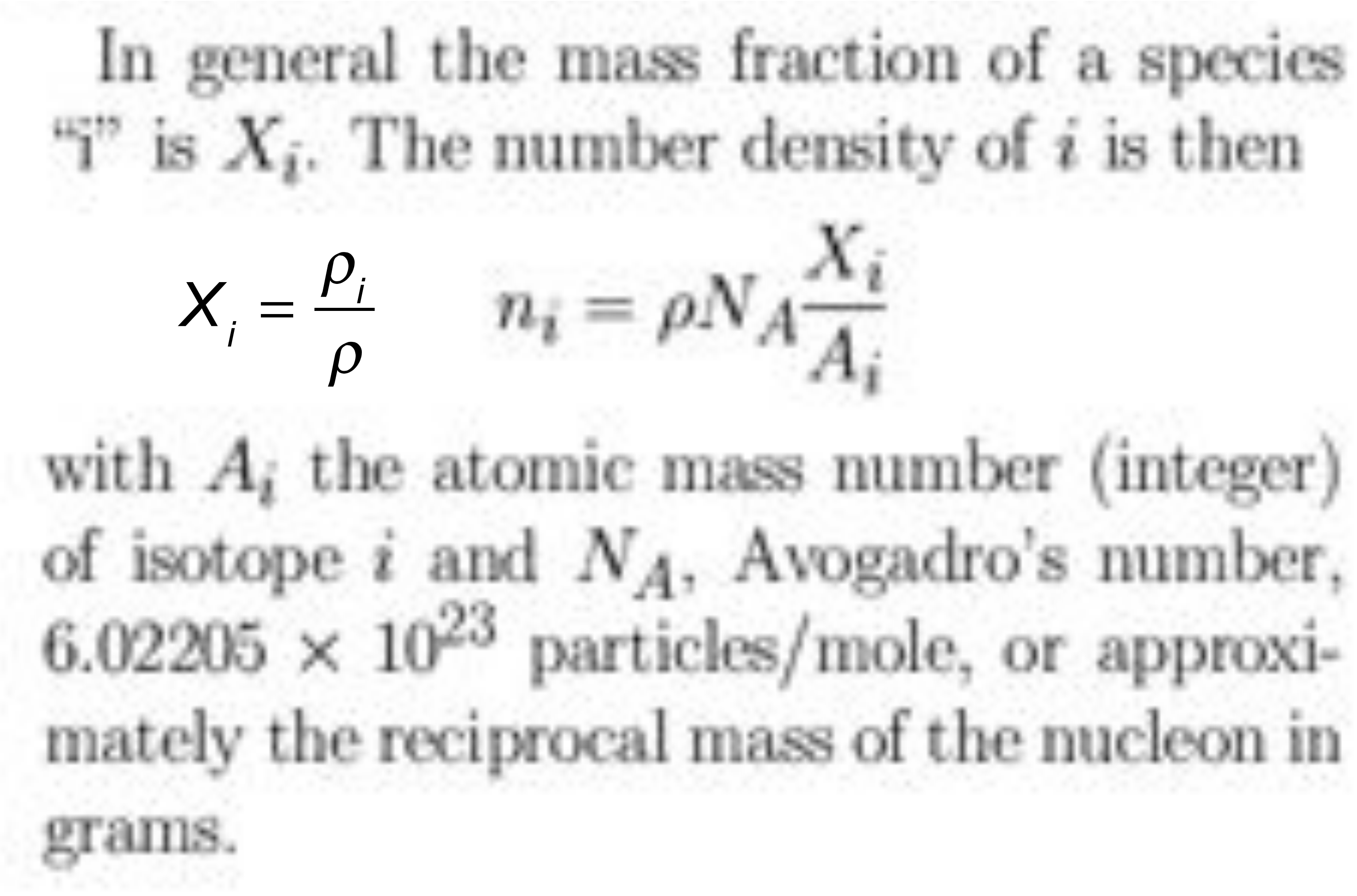
$$= 1.02 \times 10^{26} \text{ particles per cm}^3$$

continued…

$$P = n\,k\,T = \left(\frac{1.02\times10^{26}}{\mathrm{cm}^3}\right)\left(\frac{1.38\times10^{-16}\,\mathrm{erg}}{\mathrm{K}}\right)\left(10^7\ \mathrm{K}\right)$$

$$= 1.40\times10^{17}\ \ \mathrm{dyne\,cm}^{-2}$$

But there are easier ways to do the counting algebraically



In general the mass fraction of a species "i" is X_i. The number density of i is then

$$X_i = \frac{\rho_i}{\rho} \qquad n_i = \rho N_A \frac{X_i}{A_i}$$

with A_i the atomic mass number (integer) of isotope i and N_A, Avogadro's number, 6.02205×10^{23} particles/mole, or approximately the reciprocal mass of the nucleon in grams.

In this class we will extensively use the notation

$$Y_i = \frac{X_i}{A_i}$$

where Y_i is like a dimensionless number density

$$Y_i = \frac{n_i}{\rho N_A}$$

Actually the dimensions of Y are Mole/gm and N_A has dimensions particles per Mole.

Similarly we can define an electron abundance variable

$$Y_e = \frac{n_e}{\rho N_A}$$

The total gas pressure for an ideal, non-relativistic, non-degenerate ionized gas is then

$$\boxed{P_{\rm ideal} = \rho N_A kT \left[\Sigma Y_i + Y_e\right]}$$

$$Y_i = \frac{X_i}{A_i}$$

which implies

If $\mu = \left[\Sigma Y_i + Y_e\right]^{-1}$

$$P = \frac{\rho N_A kT}{\mu}$$

$$n_e = \rho N_A Y_e = \sum n_{ion} Z_{ion} = \rho N_A \sum Y_i Z_i$$

So $Y_e = \sum Y_i Z_i = \sum \frac{X_i}{A_i} Z_i$ and

$$\frac{1}{\mu} = \sum_i \frac{\left(Z_i + 1\right) X_i}{A_i}$$

$$P_{ideal} = \frac{\rho N_A kT}{\mu} \qquad \mu \approx \frac{1}{2X(H) + \frac{3}{4}X(He) + \frac{1}{2}Z}$$

with Z the mass
fraction of heavy elements.

Some examples:

1) Pure ionized H

$$Y_e = 1(1) = 1 \quad \mu = (1+1)^{-1} = \frac{1}{2}$$

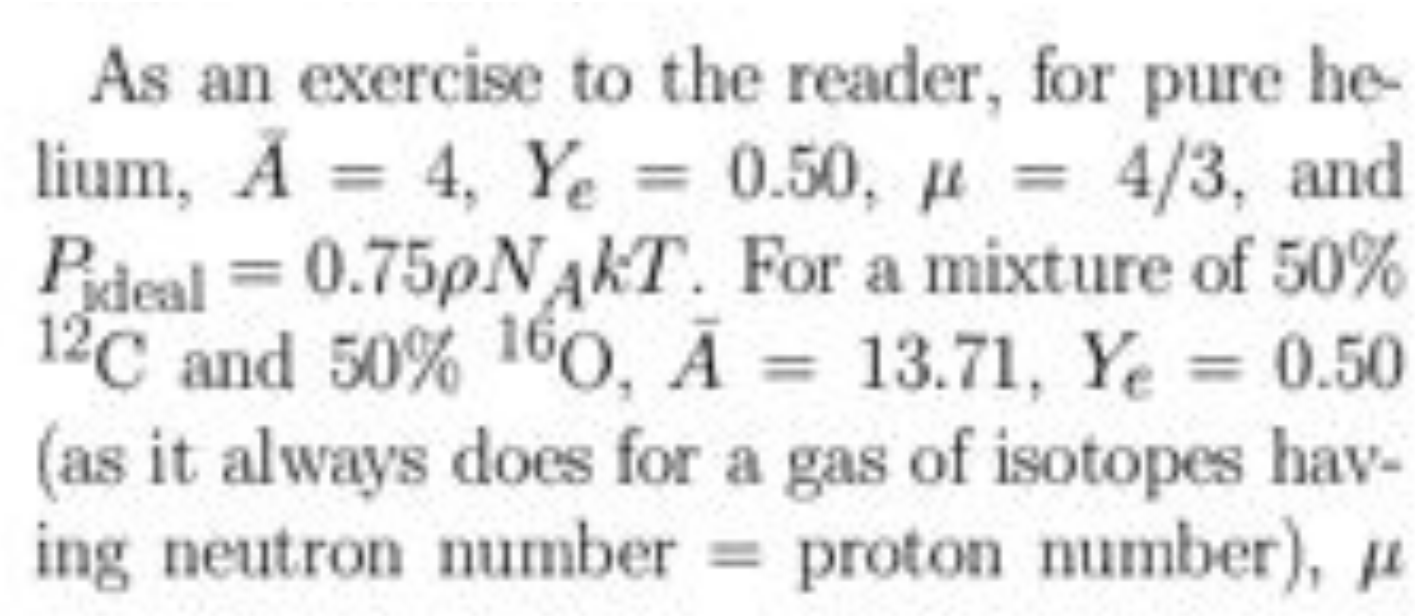
$$P_{ideal} = \left(\frac{1}{2}\right)^{-1} \rho N_A kT = 2\,\rho N_A kT$$

b) 75% H, 25% He by mass

$$
\begin{aligned}
Y_e &= 0.75 + (2)\left(\frac{0.25}{4}\right) \\
&= 0.875
\end{aligned}
$$

$$
\begin{aligned}
\mu &= \left[(1+1)(0.75) \;+\; (1+2)\left(\frac{0.25}{4}\right)\right]^{-1} \\
&= 0.5926
\end{aligned}
$$

$$
P_{\mathrm{ideal}} = 1.69\,\rho N_A kT
$$



As an exercise to the reader, for pure helium, $\bar{A} = 4$, $Y_e = 0.50$, $\mu = 4/3$, and $P_{\rm ideal} = 0.75\rho N_A kT$. For a mixture of 50% $^{12}\mathrm{C}$ and 50% $^{16}\mathrm{O}$, $\bar{A} = 13.71$, $Y_e = 0.50$ (as it always does for a gas of isotopes having neutron number = proton number), μ

These examples all assume complete ionization. If the ions are partly ionized then a Saha equation must be solved to get the electron abundance.

The internal energy *per gram*, u, is given by a similar integral to the pressure integral.

$$u = \frac{1}{\rho}\int_0^\infty \frac{dn(p)}{dp}\,\varepsilon(p)\,dp$$

$$= \frac{1}{\rho}\int_0^\infty \frac{4\pi n p^2}{(2\pi mkT)^{3/2}}\,e^{-p^2/(2mkT)}\,\frac{p^2}{2m}\,dp$$

non-relativistic

Recall the ideal gas pressure integral

$$P_{ideal} = \frac{1}{3}\int_0^\infty \frac{4\pi n p^2}{(2\pi mkT)^{3/2}}\,e^{-p^2/(2mkT)}\,p\,\frac{p}{m}\,dp$$

So that the internal energy for an ideal non-relativistic gas is just

$$u_{ideal} = \frac{3}{2}\frac{P_{ideal}}{\rho}$$

i.e., $\frac{3}{2}kT$ per particle

At relativistic energies:

$$p = \frac{mv}{\sqrt{1`-v^2/c^2}} \qquad E_{tot} = \frac{mc^2}{\sqrt{1`-v^2/c^2}}$$

$$p^2 - \frac{p^2v^2}{c^2} = m^2v^2 \quad \text{so } \mathrm{v}^2\left(m^2 + \frac{p^2}{c^2}\right) = p^2$$

$$v = \left(\frac{p^2}{m^2 + \frac{p^2}{c^2}}\right)^{1/2} = \frac{p/m}{\left(1 + p^2/(m^2c^2)\right)^{1/2}} = v$$

will use this later

$$E_{kin} = E_{tot} - mc^2 = \frac{pc^2}{v} - mc^2 = \frac{pc^2\left(1 + p^2/(m^2c^2)\right)^{1/2}}{p/m} - mc^2$$

$$= mc^2\left(1 + p^2/(m^2c^2)\right)^{1/2} - mc^2$$

$$E_{kin} = mc^2\left[\left(1 + p^2/(m^2c^2)\right)^{1/2} - 1\right]$$

It is common to consider the limiting cases of Newtonian and extreme relativistic motion.

$$\varepsilon = mc^2\left[\left(1+\frac{p^2}{m^2c^2}\right)^{1/2} - 1\right]$$

For small p this gives the usual classical result

$$\varepsilon = mc^2\left[\left(1+\frac{p^2}{m^2c^2}\right)^{1/2} - 1\right] \approx mc^2\left(1+\frac{1}{2}\frac{p^2}{m^2c^2}+\ldots-1\right) = \frac{p^2}{2m}$$

For large p >> mc, it gives

$$\varepsilon = mc^2\left[\left(1+\frac{p^2}{m^2c^2}\right)^{1/2} - 1\right] \approx mc^2\left(\frac{p^2}{m^2c^2}\right)^{1/2} = pc$$

A different expression is obtained for the internal energy density for an extremely relativistic gas,

$$u = \frac{1}{\rho}\int_0^\infty \frac{dn(p)}{dp}\,\varepsilon(p)\,dp$$

$$= \frac{1}{\rho}\int_0^\infty \frac{dn(p)}{dp}\,pc\,dp$$

and pressure

$$P_{ideal} = \frac{1}{3}\int_0^\infty \frac{dn(p)}{dp}\,p\,c\,dp$$

So that the internal energy for a relativistic gas is

$$u_{rel} = 3\frac{P_{rel}}{\rho}$$

This happens because in the energy integral pc is the energy of a particle while in the pressure integral it is also the product of momentum and speed. As one consquence, the Virial theorem is altered.

Radiation Pressure

Radiation is a strictly relativistic, quantum mechanical phenomenon. The photon is a spin 1 particle. The Bose-Einstein distribution is used to describe the particles.

$$\frac{dn(p)}{dp} = \frac{2}{h^3} \frac{1}{e^{pc/kT} - 1} 4\pi p^2 \text{ and since pc = h}\nu$$

$$dn(\nu) = \frac{8\pi}{c^3} \frac{\nu^2 d\nu}{e^{h\nu/kT} - 1}$$

This is just the $U_\nu d\nu$ we discussed in lecture 2 divided by $h\nu$, that is it is the number density of photons ($\mathrm{cm}^{-3}\ \mathrm{Hz}^{-1}$), not energy density ($\mathrm{erg\ cm}^{-3}\ \mathrm{Hz}^{-1}$)

$$\mathrm{P} = \frac{1}{3}\int_0^\infty vp\frac{dn(p)}{dp}dp = \frac{1}{3}\int_0^\infty c\frac{h\upsilon}{c}\frac{dn(\upsilon)}{d\upsilon}d\upsilon$$

$$= \frac{1}{3}\int_0^\infty c\frac{h\upsilon}{c}\frac{8\pi\nu^2}{c^3}\frac{1}{e^{h\upsilon/kT}-1}\,d\upsilon$$

Substitute $\frac{h\upsilon}{kT} = x \qquad d\upsilon = \frac{kT}{h}\,dx$

So that

$$\mathrm{P} = \frac{1}{3}\int_0^\infty c\;\frac{kTx}{c}\,\frac{8\pi}{c^3}\left(\frac{xkT}{h}\right)^2\frac{1}{e^x-1}\;\frac{kT}{h}dx$$

$$= \frac{1}{3}\frac{8\pi}{c^3}\frac{k^4}{h^3}T^4\int_0^\infty\frac{x^3}{e^x-1}dx$$

and since

$$\int_0^\infty \frac{x^3}{e^x - 1}dx = \frac{\pi^4}{15}$$

$$P_{rad} = \frac{1}{3}\frac{8\pi}{c^3}\frac{k^4}{h^3}T^4\left(\frac{\pi^4}{15}\right) = \frac{1}{3}\,aT^4$$

Where

$$a \equiv \frac{8\pi^5 k^4}{15c^3h^3} = \frac{4\sigma}{c} = 7.5658 \times 10^{-15}\ \text{erg cm}^{-3}\ \text{K}^{-4}$$

The energy density is just 3 times this

$$u_{rad} = \frac{aT^4}{\rho}\ erg\ g^{-1}$$

$$\sigma \equiv 2\frac{k^4}{h^3c^2}\left(\frac{\pi^5}{15}\right)$$

what is the number density of photons as a function of T?

$$\frac{dn(p)}{dp}=\frac{2}{h^3}\frac{1}{e^{pc/kT}-1}4\pi p^2$$

$$n=\frac{2}{h^3}\int_0^\infty \frac{4\pi p^2}{e^{pc/kT}-1}dp \qquad p=\frac{h\upsilon}{c} \qquad q=\frac{h\upsilon}{kT}$$

$$\frac{2}{h^3}\int_0^\infty \frac{4\pi(h\nu)^2}{\left(e^{h\nu/kT}-1\right)c^2}\frac{h}{c}d\nu=\frac{8\pi}{c^3}\left(\frac{kT}{h}\right)^3\int_0^\infty q^2\left(e^q-1\right)^{-1}dq$$

$=\quad 16\pi\left(\dfrac{kT}{hc}\right)^3\xi(3)$ where ξ is the Riemann zeta function

$\Gamma(3)\xi(3)$

and $\xi(3)=1.202=$"*Avery's number*"

$$n=20.3\,T^3\ \mathrm{cm}^{-3}$$

E.g. the CMBR $n=20.3*(2.7)^3=399\ \mathrm{cm}^{-3}$

Average photon energy for blackbody radiation

What is the average photon energy in BB radiation?

The total energy density per unit volume is

$$u_{rad}\rho = aT^4 = \left(\frac{8\pi^5 k^4}{15h^3c^3}\right)T^4 \ \text{erg cm}^{-3}$$

The number density of photons per unit volume is

$$n = 16\pi\left(\frac{kT}{hc}\right)^3 \xi(3) \ \text{cm}^{-3}$$

So the average energy per photon is

$$\bar{\varepsilon}_\gamma = \left(\frac{\pi^4}{(1.202)(30)}\right)kT = 2.70\ kT$$

Electron Degeneracy Pressure

So long as the electrons are “ideal”, not crowded to the extent that their quantum mechanical wavelengths become an issue (i.e. h/p negligible compared with average electron spacing), the electrons can also be treated as an ideal gas. At high density and low temperature however, this assumption breaks down.

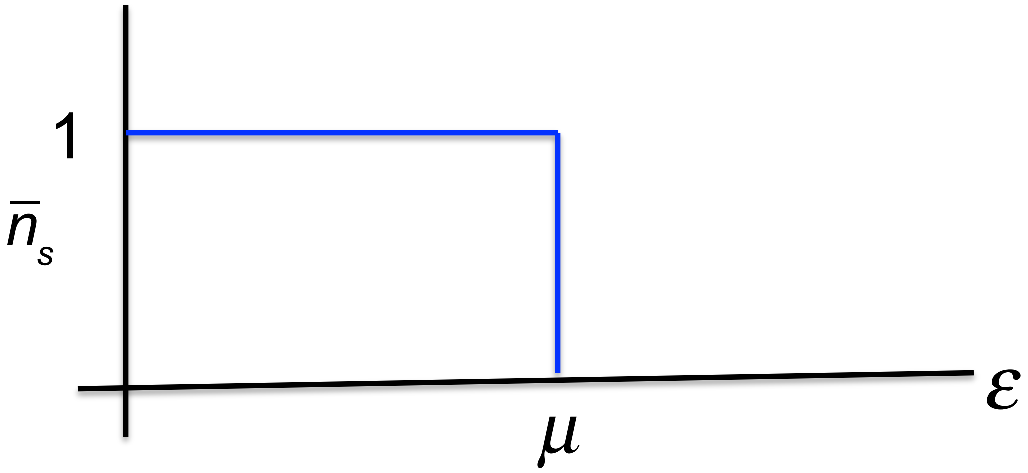
Consider the low temperature limit where the momentum of the electrons is entirely due to their quantum mechanical motion. Then quantum statistics apply not MB. The same argument applies to neutrons or protons, any particle with spin ½ (Fermions). There is a general solution that includes a mixture of degenerate and thermal motion but it involves “Fermi Integrals” and is more complicated

Electron Degeneracy Pressure

According to the Fermi-Dirac distribution function, the mean occupancy for a single particle state is

$$\bar{n}_s = \frac{1}{e^{(\varepsilon-\mu)/kT}+1}$$

At T $\approx$ 0 this takes on a value of either 0 or 1 depending upon whether $\varepsilon > \mu$ or $\varepsilon < \mu$. A plot looks like



μ here is the chemical potential, or for electrons, the "Fermi energy". It is a function of density and gives the last energy state into which electrons are forced if all the states below are as full as the uncertainty principle allows

Electron Degeneracy Pressure

In this fully degenerate limit

$$n(p)dp = \frac{2}{h^3} 4\pi p^2 \ \mathrm{dp} \qquad \mathrm{p} < \mathrm{p}_0 \qquad (\mathrm{NR}\ \ \frac{p_0^2}{2m} = \mu = \varepsilon_F)$$

$$= 0 \qquad p > p_0$$

$$n_e = \int_0^{p_0} \frac{2}{h^3} 4\pi p^2 \ \mathrm{dp} = \frac{8\pi}{3h^3} p_0^3$$

$$p_0 = \left(\frac{3h^3 n_e}{8\pi} \right)^{1/3}$$

This is the "Fermi momentum" for electron density n_e. It is the maximum energy of the degenerate electrons. The Fermi energy $\mu = \varepsilon_F = p_0^2/2m_e$.

The pressure is given by the pressure integral:

1) Non-relativistic case (fully degenerate):

$$\mathrm{P}_{\deg}^{NR} = \frac{1}{3}\int_0^\infty \frac{dn(p)}{dp}\, v\, p\, dp$$

$$= \frac{1}{3}\int_0^{p_0} \left(\frac{2}{h^3}4\pi p^2\right) p \left(\frac{p}{m}\right) dp = \left(\frac{8\pi}{3mh^3}\right)\int_0^{p_0} p^4\, dp = \left(\frac{8\pi}{3mh^3}\right)\left(\frac{p_0^5}{5}\right)$$

$$= \left(\frac{8\pi}{15mh^3}\right)\left(\frac{3h^3 n_e}{8\pi}\right)^{5/3} = \left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{20m}\, n_e^{5/3}$$

$$= \left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{20m}\left(\rho N_A Y_e\right)^{5/3} = 1.0036 \times 10^{13}\ \left(\rho Y_e\right)^{5/3}\ \text{ dyne cm}^{-2}$$

$Y_e = \sum Z_i \frac{X_i}{A_i} = \sum Z_i Y_i \equiv \frac{1}{\mu_e}$ is typically 1/2 after H burning

As n_e goes up the speed of each electron rises

$$p_0 = m_e \mathrm{v} = \left(\frac{3}{8\pi} n_e h^3 \right)^{1/3}$$

$$\mathrm{v} = \frac{1}{m_e} \left(n_e \left(\frac{3h^3}{8\pi} \right) \right)^{1/3} \qquad n_e \approx \frac{1}{2} \rho N_A$$

for elements other than H

$$\mathrm{v} = \left(\frac{3\rho N_A h^3}{16\pi m_e^3} \right)^{1/3} \approx 2 \; x \; 10^{10} \left(\frac{\rho}{10^6 \text{ gm cm}^{-3}} \right)^{1/3} \text{ cm s}^{-1}$$

At around 10^7 gm cm^{-3} the electrons will move close to the speed of light.

2) And so for the relativistic case:

$$P^{R}_{\text{deg}} = \frac{1}{3}\int_0^{\infty} \frac{dn(p)}{dp}\, v\, p\, dp$$

$$= \frac{1}{3}\int_0^{p_0} \left(\frac{2}{h^3}4\pi p^2\right) p\, c\, dp = \left(\frac{8\pi c}{3h^3}\right)\int_0^{p_0} p^3\, dp = \left(\frac{8\pi c}{3h^3}\right)\left(\frac{p_0^4}{4}\right)$$

$$= \left(\frac{2\pi}{3h^3}\right)\left(\frac{3h^3 n_e}{8\pi}\right)^{4/3} = \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3}(hc)\, n_e^{4/3}$$

$$= \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3}(hc)\left(\rho N_A Y_e\right)^{4/3} = 1.2435\times 10^{15}\ (\rho \mathrm{Y}_e)^{4/3}\ \text{dyne cm}^{-2}$$

The gas becomes relativistic when (approximately)

$$\frac{p_F^2}{2m_e} = p_F c \quad \text{or} \quad \mathrm{p}_F = 2m_e c \qquad \frac{h}{2}\left(\frac{3n_e}{\pi}\right)^{1/3} = 2m_e c$$

$$n_e = \rho N_A \mathrm{Y}_e = \frac{64\pi}{3}\left(\frac{m_e c}{h}\right)^3 = 4.69 \times 10^{30} \ \text{ gm cm}^{-3}$$

or $\rho \mathrm{Y}_e = 7.80 \times 10^6 \ \text{ g cm}^{-3}$

It becomes degenerate when (approximately)

$$\frac{\rho N_A Y_e kT}{\mu} \approx 1.24 \times 10^{15} \ \left(\rho Y_e\right)^{4/3} \qquad RD$$

or

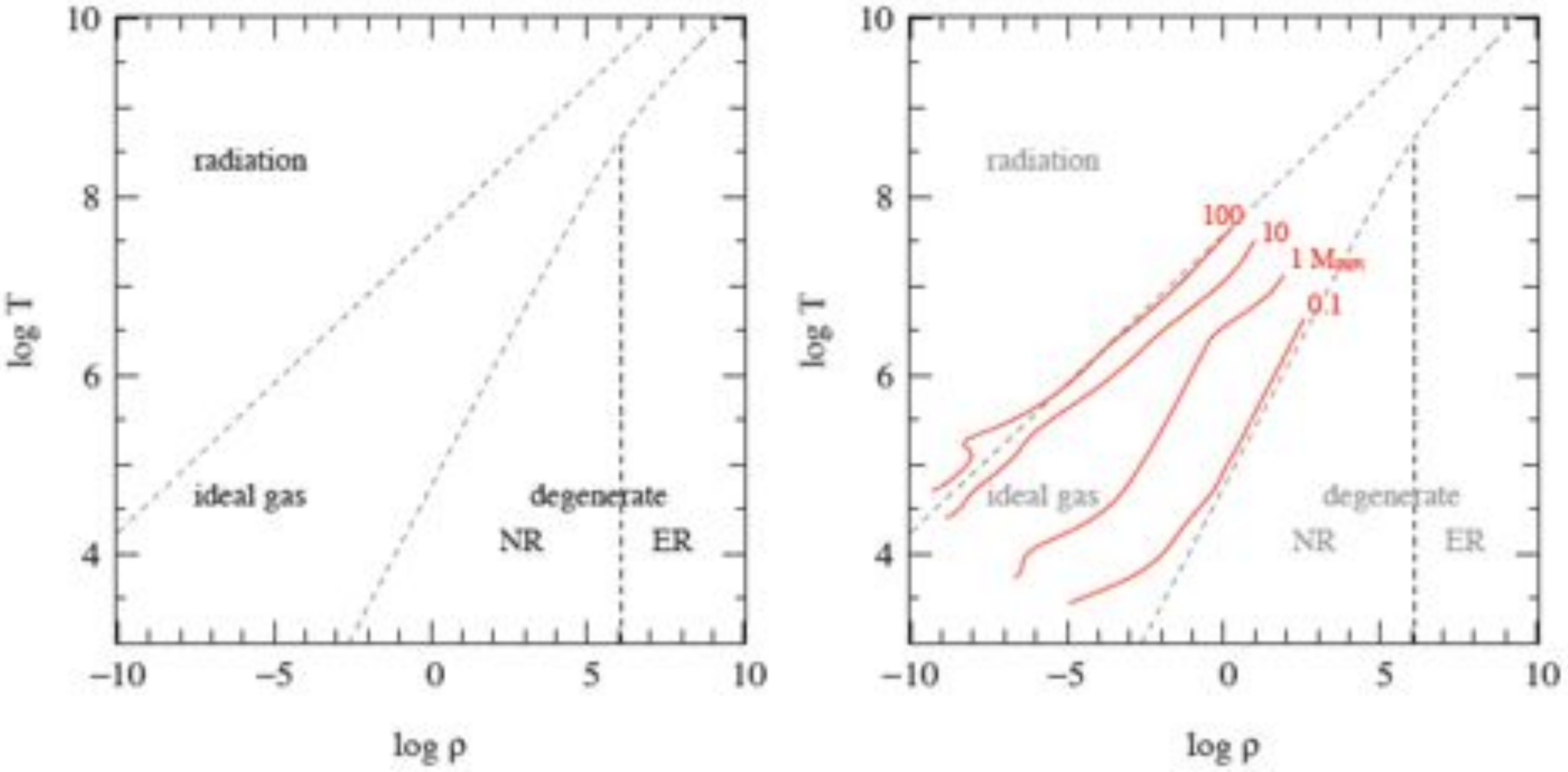
$$\frac{\rho N_A Y_e kT}{\mu} \approx 1.00 \times 10^{13} \ \left(\rho Y_e\right)^{5/3} \qquad NRD$$

This corresponds to $\dfrac{T}{\rho^{2/3}} = 1.21 \times 10^5 \mu Y_e^{5/3}$ for RD

(but $\rho Y_e > few \times 10^6$)

and $\dfrac{T}{\rho^{1/3}} = 1.50 \times 10^7 \mu Y_e^{4/3}$ g cm^{-3} for NRD

Summary figure EOS regimes – Pols Fig 3.4



See Pols p 31 for discussion

Fermi energy (extreme limits of degeneracy and relativity)

The Fermi energy is $p_0^2 / 2m_e$ for the non-relativistic case

That is $\varepsilon_F = \frac{h^2}{8m}\left(\frac{3n_e}{\pi}\right)^{2/3}$ since $p_0 = \frac{h}{2}\left(\frac{3n_e}{\pi}\right)^{1/3}$

$$= \frac{h^2}{8m}\left(\frac{3n_e}{\pi}\right)^{2/3} = \frac{h^2}{8m}\left(\frac{3\rho N_A Y_e}{\pi}\right)^{2/3}$$

$$= 4.16\times10^{-11}\left(\rho Y_e\right)^{2/3} \text{ erg} = 26\left(\rho Y_e\right)^{2/3} \text{ eV}$$

For relativistic Fermi energy, the momentum is simply multiplied by c

$$\varepsilon_F = pc = \frac{hc}{2}\left(\frac{3n_e}{\pi}\right)^{1/3} = \left(\frac{3h^3c^3\rho N_A Y_e}{8\pi}\right)^{1/3} = 8.26\times10^{-7}(\rho_6 Y_e)^{1/3} \text{ erg}$$

$= 0.516$ MeV $(\rho_6 Y_e)^{1/3}$ [*for electrons*] $\rho_6 = \left(\frac{\rho}{10^6 \text{ g cm}^{-3}}\right)$

For the general case of total degeneracy and arbitrary relativity

$$P_{\text{deg}} = \frac{1}{3}\frac{8\pi}{h^3}\int_0^{p_0} p^2 v\, p\, dp \qquad v = \frac{p/m}{\left[1+p^2/(m^2c^2)\right]^{1/2}}$$

$$= \frac{8\pi c}{3h^3}\int_0^{p_0} p^3 \frac{p/(mc)}{\left[1+p^2/(m^2c^2)\right]^{1/2}} dp$$

$$= \frac{8\pi c^5 m^4}{3h^3}\int_0^{x} \frac{\xi^4 d\xi}{\left(1+\xi^2\right)^{1/2}} \qquad \xi = p/mc \qquad x = p_0/mc$$

$$= 1.009\times 10^{-2}\left(\rho Y_e\right)^{1/3}$$

$$\int_0^{x} \frac{\xi^4 d\xi}{\left(1+\xi^2\right)^{1/2}} = \frac{1}{8}\left[x\left(2x^2-3\right)\left(1+x^2\right)^{1/2} + 3\sinh^{-1}x\right] = \frac{f(x)}{8}$$

$$P_{\text{deg}} = \frac{\pi c^5 m^4}{3h^3} f(x)$$

$$\sinh^{-1}(x) \approx x - \frac{x^3}{6} + \frac{3x^5}{40} - \frac{15x^7}{336} + \ldots.$$

for large x $\quad$ $f(x)=2x^4-2x^2+\ldots$

$$P_{\deg}=\frac{2\pi c^5 m^4}{3h^3}\left(\frac{3h^3 n_e}{8\pi m^3 c^3}\right)^{4/3}=\left(\frac{3}{\pi}\right)^{1/3}\frac{hc}{8}n_e^{4/3}$$

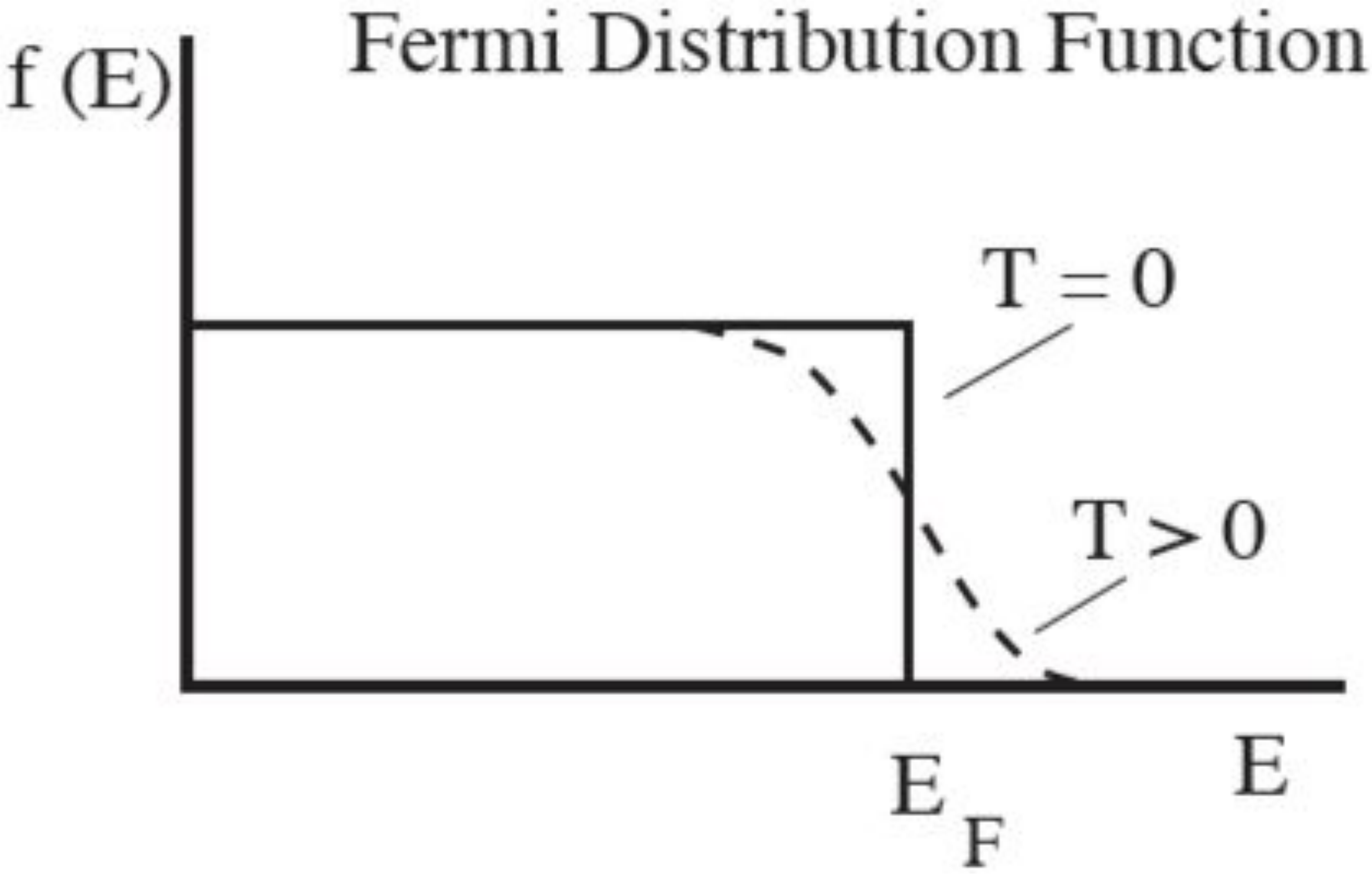
for small x $\quad$ $f(x)=\frac{8}{5}x^5-\frac{4}{7}x^7+\ldots$

$$P_{\deg}=\frac{8}{5}\frac{\pi c^5 m^4}{3h^3}\left(\frac{3h^3 n_e}{8\pi m^3 c^3}\right)^{5/3}=\frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{m}n_e^{5/3}$$

which are the extreme relativistic and non-relativistic limits found before.

$$\frac{p_0}{mc}=\frac{h}{2mc}\left(\frac{3n_e}{\pi}\right)^{1/3}=\left(\frac{3h^3 n_e}{8\pi m^3 c^3}\right)^{1/3}$$

Non-relativistic degeneneracy pressure at finite temperature



In general, for non-relativistic electrons of arbitrary degeneracy

$$n_e = \frac{8\pi}{h^3}\int_0^\infty \frac{p^2 dp}{\exp(\alpha + p^2/2mkT)+1}$$

$$\alpha \equiv -\frac{\mu}{kT}$$ (*Clayton* 2-69)

use n_e to get α

$$P_e = \frac{8\pi}{3h^3 m}\int_0^\infty \frac{p^4 dp}{\exp(\alpha + p^2/2mkT)+1}$$

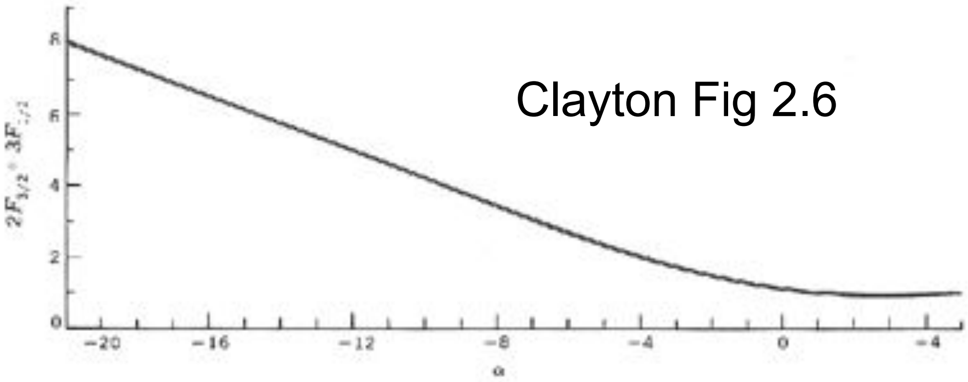
define u = $\frac{p^2}{2mkT}$ and separate out the integrals

$$n_e = \frac{4\pi}{h^3}\left(2mkT\right)^{3/2} F_{1/2}(\alpha) \qquad P_e = \frac{8\pi}{3h^3}\left(2mkT\right)^{3/2} kT\, F_{3/2}(\alpha)$$

$$F_{1/2}(\alpha) = \int_0^\infty \frac{u^{1/2} du}{\exp(\alpha + u)+1} \qquad F_{3/2}(\alpha) = \int_0^\infty \frac{u^{3/2} du}{\exp(\alpha + u)+1}$$

$$P_e = n_e kT\left(\frac{2F_{3/2}(\alpha)}{3F_{1/2}(\alpha)}\right)$$

empirically the factor in parenthesis
becomes substantially different
from unity when $\alpha < -2$



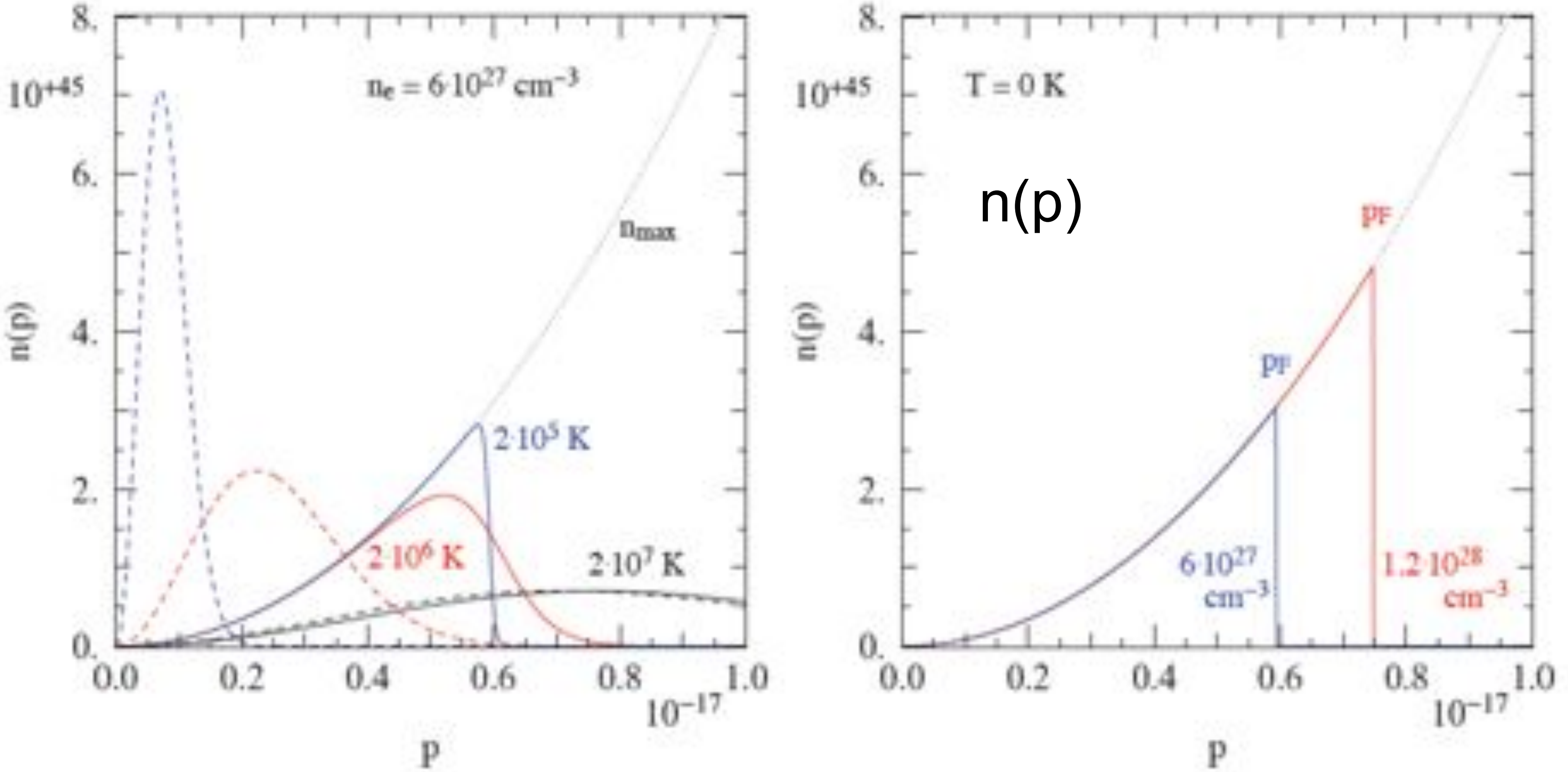
Clayton Fig 2.6

For the case of arbitrary relativity and degeneracy all modern codes use tables or expansion formulae good in various regions of temperature and density, A commonly used set of tables is the "Helmholz equation of state" by Frank Timmes.

http://cococubed.asu.edu/code_pages/eos.shtml

Please check out this site there is an enormous amount of useful interesting information there.

Pols. p 31



n(p)

Figure 3.2. *Left:* Electron momentum distributions $n(p)$ for an electron density of $n_e = 6 \times 10^{27}$ cm^{-3} (corresponding to $\rho = 2 \times 10^4$ g/cm^{-3} if $\mu_e = 2$), and for three different temperatures: $T = 2 \times 10^7$ K (black lines), 2×10^6 K (red lines) and 2×10^5 K (blue lines). The actual distributions, governed by quantum mechanics, are shown as solid lines while the Maxwell-Boltzmann distributions for the same n_e and T values are shown as dashed lines. The dotted line n_{max} is the maximum possible number distribution if all quantum states with momentum p are occupied. *Right:* Distributions in the limit $T = 0$, when all lowest available momenta are fully occupied. The blue line is for the same density as in the left panel, while the red line is for a density two times as high.

Adiabatic exponents

The first law of thermodynamics (from last time) is

$$\frac{du}{dt} + P\frac{d}{dt}\left(\frac{1}{\rho}\right) = \varepsilon - \frac{dF}{dm}$$

If no heat is being generated internally and no flux flowing in or out we have a adiabatic condition (dq = TdS =0) and

$$du + P\, d\left(\frac{1}{\rho}\right) = 0$$

But we have just seen u can typically be written as

$$u = \varphi \frac{P}{\rho} \qquad \text{with } 1.5 < \varphi < 3.0$$

for a mixture of radiation and ideal gas

So

$$\varphi \mathrm{P} d\left(\frac{1}{\rho}\right)+\varphi\frac{1}{\rho}dP+Pd\left(\frac{1}{\rho}\right)=(\varphi+1)Pd\left(\frac{1}{\rho}\right)+\varphi\frac{1}{\rho}dP=0$$

$$-(\varphi+1)P\left(\frac{d\rho}{\rho^{2}}\right)+\varphi\frac{1}{\rho}dP=0$$

multiply by $\frac{\rho}{P}$

$$-(\varphi+1)\left(\frac{d\rho}{\rho}\right)+\varphi\frac{dP}{P}=0$$

$$x=e^{\log x}$$

$$dx=d\log x\ e^{\log x}=d\log x\ x$$

$$\frac{dx}{x}=d\log x$$

$$d\log\rho=\frac{\phi}{\phi+1}d\log P$$

$$\mathrm{P} \propto \rho^{\frac{\varphi+1}{\varphi}}$$

The power in d ln P/d ln ρ is the adiabatic exponent

$$\gamma_a=\frac{\varphi+1}{\varphi}$$

which ranges from 5/3 to 4/3 for non-relativistic and relatvistic cases

$$\mathrm{P} = \mathrm{K}_a\,\rho^{\gamma_a}$$

Prialnik p. 45 spends some time discussing the effect of ionization on the adiabatic exponent. This is partly because as we shall see later, γ > 4/3 is hydrodynamically stable, but γ < 4/3 is not.

Prialnik shows that for a degree of ionization for an ideal gas

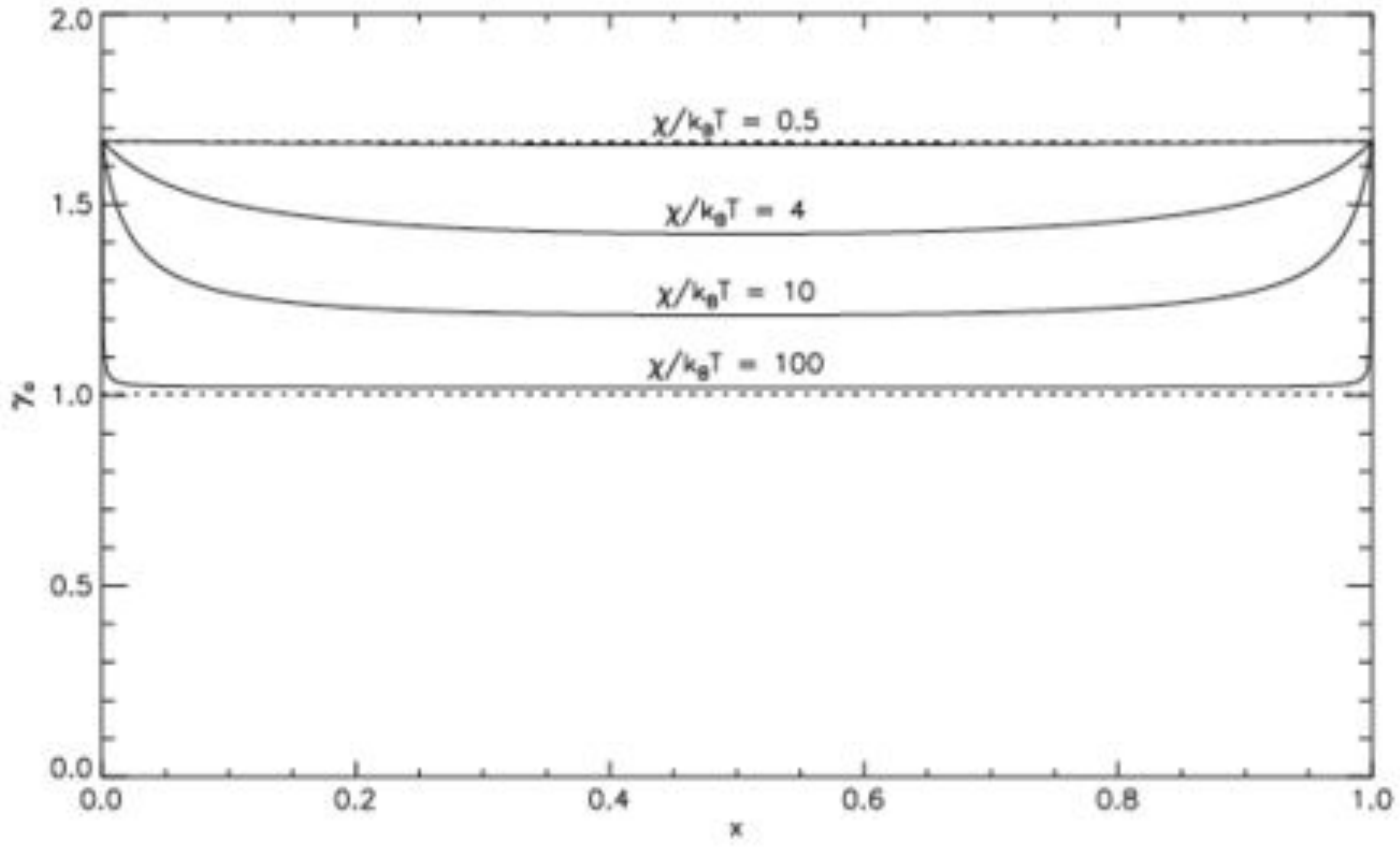
$$x = \frac{n_+}{n_0 + n_+}$$

the adiabatic exponent is

$\chi =$ ionization potential

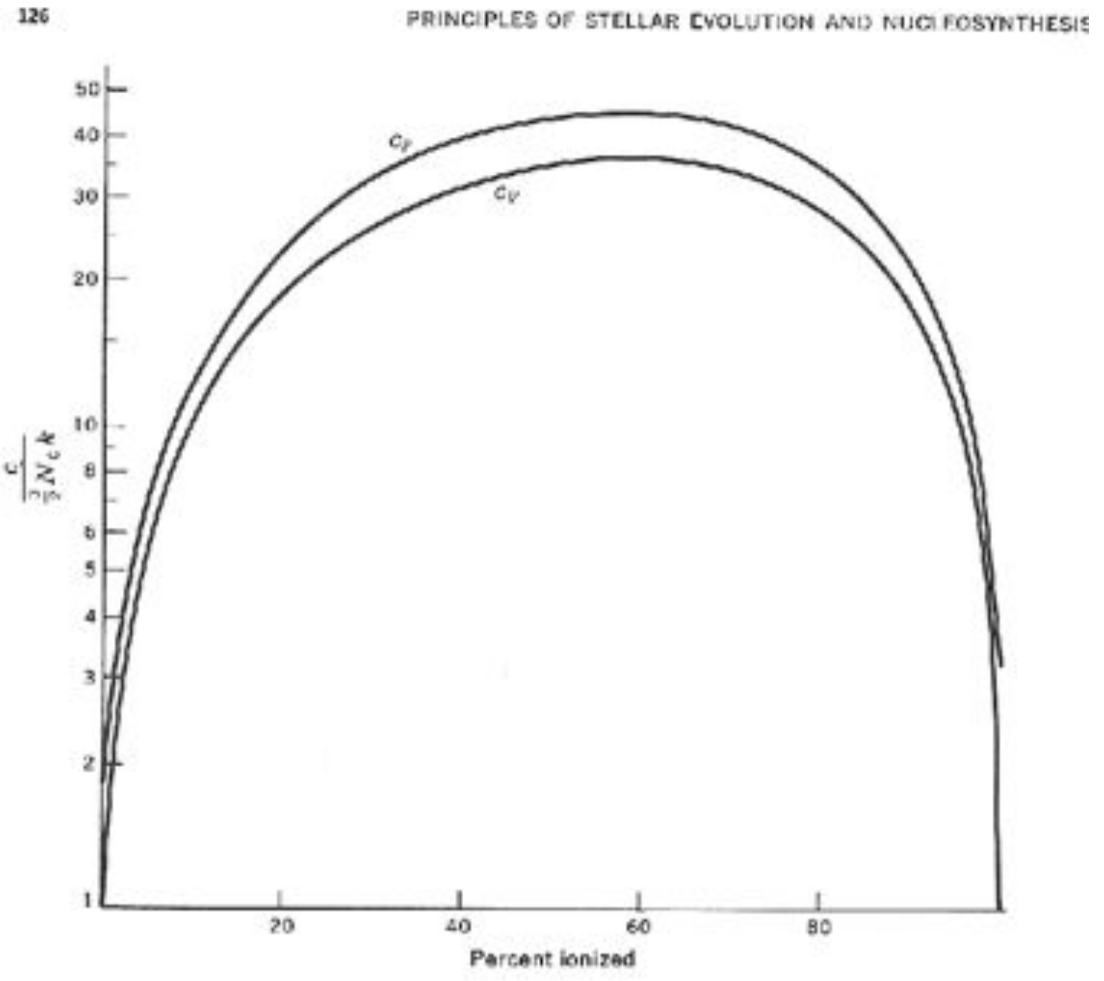
$$\gamma_a = \frac{5 + \left(\frac{5}{3} + \frac{\chi}{kT}\right)^2 x\left(1 - x\right)}{3 + \left[\frac{3}{2} + \left(\frac{3}{2} + \frac{\chi}{kT}\right)^2\right] x(1 - x)}$$

which varies from 5/3 for either x = 0 or 1 but down to values as low as 1.21 if the ionization potential, χ, is 10 kT



See Glatzmaier and Krumholz Notes 6 - B

The reason is the heat capacity. Like water approaching a phase transition there comes a temperature where most of the added energy goes into ionization (boiling) rather than raising the temperature.



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